FANTASTIC SYMMETRIES AND WHERE TO FIND THEM

Maria Clara Nucci

University of Perugia & INFN-Perugia, Italy

XXI$^{\mathrm{st}}$ International Conference
Geometry, Integrability and Quantization
June 3-8, 2019, Varna, Bulgaria

Lecture 3: Noether symmetries I.

- Hidden linearity of nonlinear equations.
- Inequivalent Lagrangians and their Noether symmetries.
- Quantization of classical mechanics problem by means of the preservation of Noether symmetries: the method.
- Quantization of classical mechanics problem by means of the preservation of Noether symmetries: examples.

Superintegrability and hidden linearity I

The Hamiltonian

$$H = \frac{1}{2}(p_1^2 + p_2^2) + \frac{\alpha u_2}{u_1^{2/3}}$$

yields a superintegrable (maximally integrable) system

$$\dot{u}_1 = p_1, \qquad \dot{u}_2 = p_2, \qquad \dot{p}_1 = \frac{2\alpha u_2}{3u_1^{5/3}}, \qquad \dot{p}_2 = -\frac{\alpha}{u_1^{2/3}}$$

since there exist two independent integrals of motion [Post and Winternitz, J.Phys.A.,2011], i.e.:

$$\begin{aligned} I_1 &= 3p_1^2 p_2 + 2p_2^3 + 9\alpha u_1^{1/3} p_1 + 6\alpha u_2 p_2/(u_1^{2/3}), \\ I_2 &= p_1^4 + 4\alpha u_2 p_1^2/(u_1^{2/3}) - 12u_1^{1/3}\alpha p_1 p_2 - 2\alpha^2(9u_1^2 - 2u_2^2)/(u_1^{4/3}). \end{aligned}$$

Then in MCN & Post, JPhysA, 2012...

The corresponding Lagrangian equations are

$$\ddot{u}_1 = \frac{2\alpha u_2}{3u_1^{5/3}}, \qquad \ddot{u}_2 = -\frac{\alpha}{u_1^{2/3}}.$$

They admits a 2-dim Lie symmetry algebra A_2 generated by

$$\Gamma_1 = \partial_t, \qquad \Gamma_2 = \frac{5}{6}t\partial_t + u_1\partial_{u_1} + u_2\partial_{u_2}.$$

The corresponding Lagrangian equations are

$$\ddot{u}_1 = \frac{2\alpha u_2}{3u_1^{5/3}}, \qquad \ddot{u}_2 = -\frac{\alpha}{u_1^{2/3}}.$$

They admits a 2-dim Lie symmetry algebra A_2 generated by

$$\Gamma_1 = \partial_t, \qquad \Gamma_2 = \frac{5}{6}t\partial_t + u_1\partial_{u_1} + u_2\partial_{u_2}.$$

Since system

$$\dot{u}_1 = p_1, \qquad \dot{u}_2 = p_2, \qquad \dot{p}_1 = \frac{2\alpha u_2}{3u_1^{5/3}}, \qquad \dot{p}_2 = -\frac{\alpha}{u_1^{2/3}}$$

is autonomous we can choose one of the dependent variables as new independent variable, namely $u_1 = y$. Then

$$u_2' = \frac{p_2}{p_1}, \qquad p_1' = \frac{2\alpha u_2}{3y^{5/3}p_1}, \qquad p_2' = -\frac{\alpha}{y^{2/3}p_1},$$

Eliminating one of the dependent variables, e.g. from the second equation $u_2 = (3y^{5/3}p_1 p_1')/(2\alpha)$ and then the system becomes:

$$p_1'' = \frac{2y^{1/3}\alpha p_2 - 5yp_1^2 p_1' - 3y^2 p_1 p_1'^2}{3y^2 p_1^2}, \qquad p_2' = -\frac{\alpha}{y^{2/3}p_1}. \qquad (1)$$

It admits a 10-dim Lie symmetry algebra isomorphic to the de Sitter algebra $o(3,2)$ generated by:

$$\begin{aligned}
X_1 &= -\frac{1}{972\alpha^2 y^{2/3}}\Big((3yp_2^8 - 324\alpha^2 y^{5/3} p_2^4 + 8748\alpha^4 y^{7/3} - 972\alpha^2 y^{5/3} p_1^2 p_2^2)\partial_y \\
&\quad + (1944\alpha^3 y p_2^2 p_1 - 108\alpha^2 y^{2/3} p_2^5 + 5832\alpha^4 y^{4/3} p_2)\partial_{p_2} + (2916\alpha^4 y^{4/3} p_1 - p_1 p_2^8 \\
&\quad + 1296\alpha^3 y p_2^3 - 24\alpha y^{1/3} p_2^7 + 432\alpha^2 y^{2/3} p_1 p_2^4 + 324\alpha^2 y^{2/3} p_1^3 p_2^2 + 1944\alpha^3 p_2 p_1^2 y)\partial_{p_1}\Big), \\
X_2 &= \frac{1}{486\alpha^2 y^{2/3}}\Big((162\alpha^2 y^{5/3} p_2^3 - 3yp_2^7 + 486\alpha^2 y^{5/3} p_2 p_1^2)\partial_y \\
&\quad + (81\alpha^2 y^{2/3} p_2^4 - 1458\alpha^4 y^{4/3} - 972\alpha^3 y p_1 p_2)\partial_{p_2} + (21\alpha y^{1/3} p_2^6 - 486\alpha^3 y p_2^2 \\
&\quad - 486\alpha^3 y p_1^2 - 270\alpha^2 y^{2/3} p_1 p_2^3 - 162\alpha^2 y^{2/3} p_2 p_1^3 + p_1 p_2^7)\partial_{p_1}\Big), \\
X_3 &= \frac{1}{243\alpha^2 y^{2/3}}\Big((243\alpha^2 y^{5/3} p_1^2 - 3yp_2^6)\partial_y + (54\alpha^2 y^{2/3} p_2^3 - 486\alpha^3 y p_1)\partial_{p_2} \\
&\quad + (p_1 p_2^6 - 81\alpha^2 y^{2/3} p_1^3 + 18\alpha y^{1/3} p_2^5 - 162\alpha^2 y^{2/3} p_1 p_2^2)\partial_{p_1}\Big), \\
X_4 &= \frac{1}{972\alpha^2 y^{2/3}}\Big((3yp_2^4 + 162 y^{5/3}\alpha^2)\partial_y + (54\alpha^2 y^{2/3} p_1 - 12 y^{1/3}\alpha p_2^3 - p_1 p_2^4)\partial_{p_1}\Big), \\
X_5 &= \frac{1}{27\alpha^2 y^{2/3}}\Big(-3yp_2^3\partial_y + 27\alpha^2 y^{2/3}\partial_{p_2} + (9p_2^2 y^{1/3}\alpha + p_1 p_2^3)\partial_{p_1}\Big),
\end{aligned}$$

$$
\begin{aligned}
X_6 &= \frac{1}{36\alpha^2 y^{2/3}}\Big((54y^{5/3}\alpha^2 - 3yp_2^4)\partial_y + 36y^{2/3}\alpha^2 p_2\partial_{p_2} \\
&\quad +(p_2^4 p_1 - 18y^{2/3}\alpha^2 p_1 + 12y^{1/3}\alpha p_2^3)\partial_{p_1}\Big), \\
X_7 &= \frac{1}{54\alpha^2 y^{2/3}}\Big((162y^{5/3}\alpha^2 p_2 - 3yp_2^5)\partial_y + 54p_2^2 y^{2/3}\alpha^2\partial_{p_2} \\
&\quad +(p_1 p_2^5 + 15y^{1/3}\alpha p_2^4 - 162\alpha^3 y - 54y^{2/3}\alpha^2 p_1 p_2)\partial_{p_1}\Big), \\
X_8 &= \frac{1}{3y^{2/3}}(3y\partial_y - p_1\partial_{p_1}), \\
X_9 &= \frac{1}{3y^{2/3}}\Big(3yp_2\partial_y - (3y^{1/3}\alpha + p_1 p_2)\partial_{p_1}\Big), \\
X_{10} &= \frac{p_2}{3y^{2/3}}\Big(3yp_2\partial_y - (6y^{1/3}\alpha + p_1 p_2)\partial_{p_1}\Big),
\end{aligned}
$$

which implies that system (1) is

$$X_6 = \frac{1}{36\alpha^2 y^{2/3}}\Big((54y^{5/3}\alpha^2 - 3yp_2^4)\partial_y + 36y^{2/3}\alpha^2 p_2\partial_{p_2}$$
$$+(p_2^4 p_1 - 18y^{2/3}\alpha^2 p_1 + 12y^{1/3}\alpha p_2^3)\partial_{p_1}\Big),$$
$$X_7 = \frac{1}{54\alpha^2 y^{2/3}}\Big((162y^{5/3}\alpha^2 p_2 - 3yp_2^5)\partial_y + 54p_2^2 y^{2/3}\alpha^2\partial_{p_2}$$
$$+(p_1 p_2^5 + 15y^{1/3}\alpha p_2^4 - 162\alpha^3 y - 54y^{2/3}\alpha^2 p_1 p_2)\partial_{p_1}\Big),$$
$$X_8 = \frac{1}{3y^{2/3}}(3y\partial_y - p_1\partial_{p_1}),$$
$$X_9 = \frac{1}{3y^{2/3}}\Big(3yp_2\partial_y - (3y^{1/3}\alpha + p_1p_2)\partial_{p_1}\Big),$$
$$X_{10} = \frac{p_2}{3y^{2/3}}\Big(3yp_2\partial_y - (6y^{1/3}\alpha + p_1p_2)\partial_{p_1}\Big),$$

which implies that system (1) is linearizable!!!

Here is system (2) again:

$$p_1'' = \frac{2y^{1/3}\alpha p_2 - 5yp_1^2p_1' - 3y^2p_1p_1'^2}{3y^2p_1^2}, \qquad p_2' = -\frac{\alpha}{y^{2/3}p_1}.$$

From the second equation $p_1 = -\alpha/(y^{2/3}p_2')$ yields

$$p_2''' = \frac{1}{9\alpha^2 p_2' y^2}\left(6y^{1/3}p_2p_2'^5y^2 + 4\alpha^2p_2'^2 + 9\alpha^2p_2'p_2''y + 27\alpha^2p_2''^2y^2\right).$$

This equation admits a 7-dim Lie symmetry algebra generated by

$$Y_1 = -\frac{y^{1/3}p_2^3}{9\alpha^2}\partial_y + \partial_{p_2}, \; Y_2 = \frac{-y^{1/3}p_2^5+54\alpha^2yp_2}{54\alpha^2}\partial_y + \frac{p_2^2}{3}\partial_{p_2}, \; Y_3 = \frac{y^{1/3}p_2^4+54\alpha^2y}{54\alpha^2}\partial_y,$$

$$Y_4 = -\frac{y^{1/3}p_2^4}{9\alpha^2}\partial_y + p_2\partial_{p_2}, \; Y_5 = y^{1/3}\partial_y, \; Y_6 = y^{1/3}p_2\partial_y, Y_7 = y^{1/3}p_2^2\partial_y.$$

Thus it is linearizable (10-dim Lie algebra are its contact symm.)

$\tilde{y} = p_2, \; \tilde{p}_2 = \frac{3}{2}y^{2/3} + \frac{1}{36\alpha^2}p_2^4 \quad \Rightarrow$

Here is system (2) again:

$$p_1'' = \frac{2y^{1/3}\alpha p_2 - 5yp_1^2p_1' - 3y^2p_1p_1'^2}{3y^2p_1^2}, \qquad p_2' = -\frac{\alpha}{y^{2/3}p_1}.$$

From the second equation $p_1 = -\alpha/(y^{2/3}p_2')$ yields

$$p_2''' = \frac{1}{9\alpha^2p_2'y^2}\left(6y^{1/3}p_2p_2'^5y^2 + 4\alpha^2p_2'^2 + 9\alpha^2p_2'p_2''y + 27\alpha^2p_2''^2y^2\right).$$

This equation admits a 7-dim Lie symmetry algebra generated by

$$Y_1 = -\frac{y^{1/3}p_2^3}{9\alpha^2}\partial_y + \partial_{p_2},\ Y_2 = \frac{-y^{1/3}p_2^5+54\alpha^2yp_2}{54\alpha^2}\partial_y + \frac{p_2^2}{3}\partial_{p_2},\ Y_3 = \frac{y^{1/3}p_2^4+54\alpha^2y}{54\alpha^2}\partial_y,$$
$$Y_4 = -\frac{y^{1/3}p_2^4}{9\alpha^2}\partial_y + p_2\partial_{p_2},\ Y_5 = y^{1/3}\partial_y,\ Y_6 = y^{1/3}p_2\partial_y,\ Y_7 = y^{1/3}p_2^2\partial_y.$$

Thus it is linearizable (10-dim Lie algebra are its contact symm.)

$$\tilde{y} = p_2,\ \tilde{p}_2 = \frac{3}{2}y^{2/3} + \frac{1}{36\alpha^2}p_2^4 \quad \Rightarrow \quad \frac{\mathrm{d}^3\tilde{p}_2}{\mathrm{d}\tilde{y}^3} = 0$$

Since $y = u_1$ and $u_2 = (3y^{5/3}p_1p_1')/(2\alpha)$ then the 10-dim Lie symmetry algebra is obtained:

$$
\begin{aligned}
\Gamma_1 \quad = \quad & V_1\partial_t + \frac{1}{972\alpha^3 u_1^{4/3}}\Big((324u_1^{7/3}\alpha^3 p_2^4 - 3u_1^{5/3}\alpha p_2^8 - 8748u_1^3\alpha^5 + 972u_1^{7/3}\alpha^3 p_2^2 p_1^2)\partial_{u_1} \\
& +(5832u_1^{8/3}p_2^2\alpha^4 - 252\alpha^2 u_1^2 p_2^6 - 5832\alpha^5 u_1^2 u_2 - u_1^{4/3}p_2^8 p_1^2 - 24\alpha u_1^{5/3}p_2^7 p_1 \\
& -2\alpha u_1^{2/3}p_2^8 u_2 - 3888u_1^{5/3}p_2\alpha^4 u_2 p_1 - 1296u_1^{2/3}p_2^2\alpha^4 u_2^2 - 648u_1^{4/3}p_2^2\alpha^3 u_2 p_1^2 \\
& -648u_1^{4/3}p_2^4\alpha^3 u_2 + 1944u_1^{7/3}p_2^3\alpha^3 p_1)\partial_{u_2} \\
& +(24u_1\alpha^2 p_2^7 - 2916u_1^2\alpha^5 p_1 + u_1^{2/3}\alpha p_2^8 p_1 - 1296u_1^{5/3}\alpha^4 p_2^3 \\
& -432u_1^{4/3}\alpha^3 p_2^4 p_1 - 324u_1^{4/3}\alpha^3 p_2^2 p_1^3 - 1944u_1^{5/3}\alpha^4 p_2 p_1^2)\partial_{p_1} \\
& +(108p_2^5 u_1^{4/3}\alpha^3 - 5832p_2 u_1^2\alpha^5 - 1944p_2^2 u_1^{5/3}\alpha^4 p_1)\partial_{p_2}\Big)\,, \\
\Gamma_2 \quad = \quad & V_2\partial_t + \frac{1}{486\alpha^3 u_1^{4/3}}\Big((-3u_1^{5/3}p_2^7\alpha + 162u_1^{7/3}p_2^3\alpha^3 + 486u_1^{7/3}p_2\alpha^3 p_1^2)\partial_{u_1} \\
& +(972u_1^{7/3}p_2^2\alpha^3 p_1 - 21\alpha u_1^{5/3}p_2^6 p_1 - 2\alpha u_1^{2/3}p_2^7 u_2 + 1458u_1^{8/3}\alpha^4 p_2 \\
& -972u_1^{5/3}\alpha^4 u_2 p_1 - 648u_1^{2/3}\alpha^4 u_2^2 p_2 - 189p_2^5\alpha^2 u_1^2 - u_1^{4/3}p_2^7 p_1^2 \\
& -324u_1^{4/3}p_2\alpha^3 u_2 p_1^2 - 432u_1^{4/3}p_2^3\alpha^3 u_2)\partial_{u_2} \\
& +(u_1^{2/3}\alpha p_2^7 p_1 + 21u_1\alpha^2 p_2^6 - 486u_1^{5/3}\alpha^4 p_2^2 - 486u_1^{5/3}\alpha^4 p_1^2 - 270u_1^{4/3}\alpha^3 p_2^3 p_1 \\
& -162u_1^{4/3}\alpha^3 p_2 p_1^3)\partial_{p_1} + (81u_1^{4/3}\alpha^3 p_2^4 - 1458u_1^2\alpha^5 - 972u_1^{5/3}\alpha^4 p_2 p_1)\partial_{p_2}\Big)\,,
\end{aligned}
$$

$$
\begin{aligned}
\Gamma_3 \quad = \quad & V_3\partial_t + \frac{1}{243\alpha^3 u_1^{4/3}}\Big((243u_1^{7/3}\alpha^3 p_1^2 - 3u_1^{5/3}\alpha p_2^6)\partial_{u_1} \\
& -(18\alpha u_1^{5/3}p_2^5 p_1 + 2\alpha u_1^{2/3}p_2^6 u_2 + 324u_1^{2/3}u_2^2\alpha^4 + 135p_2^4\alpha^2 u_1^2 + u_1^{4/3}p_2^6 p_1^2 \\
& +162u_1^{4/3}\alpha^3 u_2 p_1^2 + 324u_1^{4/3}\alpha^3 u_2 p_2^2 - 486u_1^{7/3}\alpha^3 p_1 p_2)\partial_{u_2} \\
& +(u_1^{2/3}\alpha p_2^6 p_1 - 81u_1^{4/3}\alpha^3 p_1^3 + 18u_1\alpha^2 p_2^5 - 162u_1^{4/3}\alpha^3 p_2^2 p_1)\partial_{p_1} \\
& +(54u_1^{4/3}\alpha^3 p_2^3 - 486u_1^{5/3}\alpha^4 p_1)\partial_{p_2}\Big)\,, \\
\Gamma_4 \quad = \quad & V_4\partial_t + \frac{1}{972\alpha^3 u_1^{4/3}}\Big((3u_1^{5/3}\alpha p_2^4 + 162u_1^{7/3}\alpha^3)\partial_{u_1} \\
& +(12\alpha u_1^{5/3}p_2^3 p_1 + 2\alpha u_1^{2/3}p_2^4 u_2 + 54p_2^2\alpha^2 u_1^2 + u_1^{4/3}p_2^4 p_1^2 + 216u_1^{4/3}\alpha^3 u_2)\partial_{u_2} \\
& +(54u_1^{4/3}\alpha^3 p_1 - u_1^{2/3}\alpha p_2^4 p_1 - 12u_1\alpha^2 p_2^3)\partial_{p_1}\Big)\,, \\
\Gamma_5 \quad = \quad & V_5\partial_t - \frac{u_1^{1/3}p_2^3}{9\alpha^2}\,\partial_{u_1} - \frac{1}{27\alpha^3 u_1^{4/3}}\Big((9p_2^2\alpha u_1^{5/3}p_1 + 2p_2^3\alpha u_1^{2/3}u_2 \\
& +27p_2\alpha^2 u_1^2 + p_2^3 u_1^{4/3}p_1^2)\partial_{u_2} - (p_2^3 u_1^{2/3}\alpha p_1 + 9p_2^2 u_1\alpha^2)\partial_{p_1}\Big) + \partial_{p_2}\,, \\
\Gamma_6 \quad = \quad & V_6\partial_t + \frac{1}{36\alpha^3 u_1^{4/3}}\Big((54u_1^{7/3}\alpha^3 - 3u_1^{5/3}\alpha p_2^4)\partial_{u_1} \\
& -(12\alpha u_1^{5/3}p_2^3 p_1 + 2\alpha u_1^{2/3}p_2^4 u_2 + 54p_2^2\alpha^2 u_1^2 + u_1^{4/3}p_2^4 p_1^2)\partial_{p_2} \\
& +(u_1^{2/3}\alpha p_2^4 p_1 - 18u_1^{4/3}\alpha^3 p_1 + 12u_1\alpha^2 p_2^3)\partial_{p_1}\Big) + p_2\partial_{p_2}\,, \\
\Gamma_7 \quad = \quad & V_7\partial_t + \frac{1}{54\alpha^3 u_1^{4/3}}\Big((162u_1^{7/3}p_2\alpha^3 - 3u_1^{5/3}p_2^5\alpha)\partial_{u_1} \\
& -(15p_2^4\alpha u_1^{5/3}p_1 + 2p_2^5\alpha u_1^{2/3}u_2 + 90p_2^3\alpha^2 u_1^2 + u_1^{4/3}p_2^5 p_1^2)\partial_{u_2} \\
& +(u_1^{2/3}\alpha p_2^5 p_1 + 15u_1\alpha^2 p_2^4 - 162u_1^{5/3}\alpha^4 - 54u_1^{4/3}\alpha^3 p_2 p_1)\partial_{p_1}\Big) + p_2^2\partial_{p_2}\,,
\end{aligned}
$$

$$\Gamma_8 = V_8\partial_t + \frac{1}{3\alpha u_1^{2/3}}\left(3\alpha u_1\partial_{u_1} + (2u_2\alpha + u_1^{2/3}p_1^2)\partial_{u_2} - \alpha p_1\partial_{p_1}\right),$$

$$\Gamma_9 = V_9\partial_t + \frac{1}{3\alpha u_1^{2/3}}\Big(3\alpha u_1 p_2\partial_{u_1} + (3\alpha p_1 u_1 + 2\alpha u_2 p_2 + u_1^{2/3}p_2 p_1^2)\partial_{u_2}$$
$$-(\alpha p_2 p_1 + 3u_1^{1/3}\alpha^2)\partial_{p_1}\Big),$$

$$\Gamma_{10} = V_{10}\partial_t + u_1^{1/3}p_2^2\partial_{u_1} + \frac{1}{3\alpha u_1^{4/3}}\Big((6\alpha p_1 p_2 u_1^{5/3} + 2\alpha p_2^2 u_1^{2/3}u_2$$
$$+9\alpha^2 u_1^2 + p_1^2 p_2^2 u_1^{4/3})\partial_{u_2} - \alpha p_2(6\alpha u_1 + p_1 p_2 u_1^{2/3})\partial_{p_1}\Big).$$

Each $V_k = V_k(t, u_1, u_2, p_1, p_2), (k = 1, \dots, 10)$, satisfies:

$$486u_1^{5/3}\alpha^2 V_y + 486u_1^{5/3}\alpha^2 p_1 V_{u_1} + 486u_1^{5/3}\alpha^2 p_2 V_{u_2} + 324\alpha^3 u_2 V_{p_1} - 486\alpha^3 u_1 V_{p_2}$$
$$+c_1(u_1 p_2^8 - 648u_1\alpha^3 u_2 p_2^2 + 5832u_1^{7/3}\alpha^4 - 648u_1^{5/3}\alpha^2 p_2^2 p_1^2 - 378u_1^{5/3}\alpha^2 p_2^4)$$
$$-c_2(648u_1\alpha^3 u_2 p_2 + 2u_1 p_2^7 + 648u_1^{5/3}\alpha^2 p_2 p_1^2 + 432u_1^{5/3}\alpha^2 p_2^3)$$
$$+c_3(4u_1 p_2^6 - 648u_1\alpha^3 u_2 - 648u_1^{5/3}\alpha^2 p_1^2 - 324u_1^{5/3}\alpha^2 p_2^2)$$
$$-c_4(u_1 p_2^4 + 54u_1^{5/3}\alpha^2) + 36u_1 p_2^3 c_5 + c_6(27u_1 p_2^4 - 972u_1^{5/3}\alpha^2)$$
$$+c_7(18u_1 p_2^5 - 1944u_1^{5/3}\alpha^2 p_2) - 324u_1\alpha^2 c_8 - 324\alpha^2 p_2 u_1 c_9 - 324u_1\alpha^2 p_2^2 c_{10} = 0$$

with $c_k = 1$ and all the other $c_{j\neq k} = 0, (j = 1, \dots, 10)$.

We have 330 matrices

$$M_{n,m,k,j} = \det \begin{bmatrix} 1 & p_1 & p_2 & \frac{2\alpha u_2}{3u_1^{5/3}} & -\frac{\alpha}{u_1^{2/3}} \\ V_n & G_{1,n} & G_{2,n} & G_{3,n} & G_{4,n} \\ V_m & G_{1,m} & G_{2,m} & G_{3,m} & G_{4,m} \\ V_k & G_{1,k} & G_{2,k} & G_{3,k} & G_{4,k} \\ V_j & G_{1,j} & G_{2,j} & G_{3,j} & G_{4,j} \end{bmatrix},$$

where $(n, m, k, j = 1, \ldots, 10)$. Since we are interested in autonomous first integrals we only need to consider the determinant of any the 120 matrices

$$M_{n,m,k} = \det \begin{bmatrix} p_1 & p_2 & \frac{2\alpha u_2}{3u_1^{5/3}} & -\frac{\alpha}{u_1^{2/3}} \\ G_{1,n} & G_{2,n} & G_{3,n} & G_{4,n} \\ G_{1,m} & G_{2,m} & G_{3,m} & G_{4,m} \\ G_{1,k} & G_{2,k} & G_{3,k} & G_{4,k} \end{bmatrix},$$

where $(n, m, k = 1, \ldots, 10)$ are all the possible combinations without repetition of the three indices and the generators of each Lie symmetry.

We present some of those first integrals, e.g.:

$$M_{5,8,10} = 2\alpha \left(\frac{1}{2}(p_1^2 + p_2^2) + \frac{\alpha u_2}{u_1^{2/3}} \right) = 2\alpha H$$

that is the Hamiltonian H;

We present some of those first integrals, e.g.:

$$M_{5,8,10} = 2\alpha\left(\frac{1}{2}(p_1^2 + p_2^2) + \frac{\alpha u_2}{u_1^{2/3}}\right) = 2\alpha H$$

that is the Hamiltonian H;

$$M_{5,6,8} = \frac{2}{3\alpha}\left(\frac{1}{2}(p_1^2 + p_2^2) + \frac{\alpha u_2}{u_1^{2/3}}\right)^2 = \frac{2}{3\alpha}H^2$$

that is the square of the Hamiltonian H and a polynomial of order 4 in p_1 and p_2. Also the two known independent first integrals can be obtained:

We present some of those first integrals, e.g.:

$$M_{5,8,10} = 2\alpha \left(\frac{1}{2}(p_1^2 + p_2^2) + \frac{\alpha u_2}{u_1^{2/3}} \right) = 2\alpha H$$

that is the Hamiltonian H;

$$M_{5,6,8} = \frac{2}{3\alpha} \left(\frac{1}{2}(p_1^2 + p_2^2) + \frac{\alpha u_2}{u_1^{2/3}} \right)^2 = \frac{2}{3\alpha} H^2$$

that is the square of the Hamiltonian H and a polynomial of order 4 in p_1 and p_2. Also the two known independent first integrals can be obtained:

$M_{7,8,9} = -\frac{\alpha}{3} \left(3p_1^2 p_2 + 2p_2^3 + 9\alpha u_1^{1/3} p_1 + 6\alpha \frac{u_2 p_2}{u_1^{2/3}} \right) = -\frac{\alpha}{3} I_1,$

$M_{7,8,10} = -\frac{\alpha}{2} \left(12 u_1^{1/3} \alpha p_1 p_2 + 18\alpha^2 u_1^{2/3} + p_2^4 + 2p_3^2 p_2^2 + 4\alpha \frac{u_2 p_2^2}{u_1^{2/3}} \right) = \frac{\alpha}{2}(I_2 - 4H^2)$

Since Γ_1 contains powers of p_1 up to order 3 and powers of p_2 up to order 8, Γ_2 up to order 3 and 7, respectively, and Γ_3 up to order 3 and 6, respectively, then the following first integral

Since Γ_1 contains powers of p_1 up to order 3 and powers of p_2 up to order 8, Γ_2 up to order 3 and 7, respectively, and Γ_3 up to order 3 and 6, respectively, then the following first integral

$$
\begin{aligned}
M_{1,2,3} = & \frac{1}{1417176\alpha^3 u_1^{8/3}} \Big(78732 u_1^4 \alpha^4 p_2^{10} + 34012224 u_1^{8/3} \alpha^9 u_2^3 + u_1^{8/3} p_2^{18} \\
& + 4251528 u_1^{14/3} \alpha^6 p_2^6 + 27 p_1^6 u_1^{8/3} p_2^{12} + 27 u_1^{8/3} p_2^{14} p_1^4 + 486 u_1^{10/3} p_2^{14} \alpha^2 + 9 u_1^{8/3} p_2^{16} p_1^2 \\
& + 108 p_1 \alpha u_1^3 p_2^{15} + 648 p_1^3 \alpha u_1^3 p_2^{13} + 866052 p_1^2 \alpha^4 u_1^4 p_2^8 + 972 p_1^5 \alpha u_1^3 p_2^{11} + 18 u_1^2 p_2^{16} \alpha u_2 \\
& + 29160 u_1^2 p_2^{10} \alpha^4 u_2^2 + 1889568 u_1^2 \alpha^7 p_2^4 u_2^3 + 25509168 u_1^4 \alpha^7 p_2^4 u_2 + 216 u_1^{2/3} p_2^{12} \alpha^3 u_2^3 \\
& + 6804 u_1^{8/3} p_2^{12} \alpha^3 u_2 + 34992 u_1^{11/3} p_2^{11} \alpha^3 p_1 + 2204496 u_1^{8/3} p_2^6 \alpha^6 u_2^2 \\
& + 6804 u_1^{10/3} p_1^2 \alpha^2 p_2^{12} + 151632 u_1^{10/3} p_1^2 \alpha^2 p_2^{12} + 151632 u_1^{11/3} p_1^3 \alpha^3 p_2^9 \\
& + 16038 u_1^{10/3} p_1^4 \alpha^2 p_2^{10} + 2834352 p_1 \alpha^5 u_1^{13/3} p_2^7 + 108 u_1^{4/3} p_2^{14} \alpha^2 u_2^2 \\
& + 34992 u_1^{4/3} p_2^8 u_2^3 \alpha^5 + 787320 u_1^{10/3} \alpha^5 p_2^8 u_2 + 51018336 u_1^{10/3} \alpha^8 u_2^2 p_2^2 \\
& + 279936 p_1 \alpha^4 u_1^3 u_2 p_2^9 + 11337408 p_1 \alpha^7 u_1^3 u_2^2 p_2^3 + 209952 p_1^3 \alpha^4 u_1^3 u_2 p_2^7 \\
& + 108 p_1^2 \alpha u_1^2 p_2^{14} u_2 + 162 p_1^4 \alpha u_1^2 p_2^{12} u_2 + 34992 p_1^2 \alpha^4 u_1^2 p_2^8 u_2^2 \\
& + 324 p_1^2 \alpha^2 u_1^{4/3} p_2^{12} u_2^2 + 419904 p_1 \alpha^5 u_1^{7/3} u_2^2 p_2^7 + 3888 p_1 \alpha^3 u_1^{5/3} p_2^{11} u_2^2 \\
& + 3888 p_1 \alpha^3 u_1^{5/3} p_2^{11} \alpha^5 u_1^{7/3} u_2^2 p_2^7 + 11337408 p_1 \alpha^6 u_1^{11/3} u_2 p_2^5 \\
& + 3888 p_1^3 \alpha^2 u_1^{7/3} p_2^{11} u_2 + 46656 p_1^2 \alpha^3 u_1^{8/3} p_2^{10} u_2 + 2204496 p_1^2 \alpha^5 u_1^{10/3} p_2^6 u_2 \\
& + 1296 p_1 \alpha^2 u_1^{7/3} p_2^{13} u_2 + 944784 u_1^{8/3} \alpha^6 p_2^4 p_1^2 u_2^2 + 8748 u_1^{8/3} \alpha^3 p_2^8 p_1^4 u_2 \Big)
\end{aligned}
$$

is polynomial of order 6 and 18, respectively, in the components p_1 and p_2 of the momenta.
Many other polynomials first integrals of lesser order in p_1, p_2 can be derived, e.g. $M_{1,2,4}$ is a polynomial of degree 6 and 16 in p_1, p_2, respectively; $M_{1,2,5}$ is a polynomial of degree 6 and 15; $M_{1,2,6}$ and $M_{1,3,7}$ are also polynomials of degree 6 and 16; $M_{1,3,5}$, $M_{1,4,7}$ and $M_{2,3,6}$ are polynomials of degree 6 and 14; and so on.

Superintegrability and hidden linearity II

In *Friš, Mandrosov, Smorodinsky, Uhlìř and Winternitz, Phys.Lett. A, 1965* the following Hamiltonians were considered

$$H = \frac{1}{2}\left(p_1^2 + p_2^2\right) + V(x_1, x_2), \quad H = \frac{1}{2}\left(p_r^2 + \frac{p_\varphi^2}{r^2}\right) + V(r, \varphi),$$

and it was proven that only four independent potentials exist such that two quadratical integrals of motion exist.

Superintegrability and hidden linearity II

In *Friš, Mandrosov, Smorodinsky, Uhlìř and Winternitz, Phys.Lett. A, 1965* the following Hamiltonians were considered

$$H = \frac{1}{2}\left(p_1^2 + p_2^2\right) + V(x_1, x_2), \quad H = \frac{1}{2}\left(p_r^2 + \frac{p_\varphi^2}{r^2}\right) + V(r, \varphi),$$

and it was proven that only four independent potentials exist such that two quadratical integrals of motion exist.
In *G.Gubbiotti & MCN, J.Math.Phys., 2017* we have proven that the corresponding Lagrange equations can be linearized through hidden symmetries.

Superintegrability and hidden linearity II

In *Friš, Mandrosov, Smorodinsky, Uhlìř and Winternitz, Phys.Lett. A, 1965* the following Hamiltonians were considered

$$H = \frac{1}{2}\left(p_1^2 + p_2^2\right) + V(x_1, x_2), \quad H = \frac{1}{2}\left(p_r^2 + \frac{p_\varphi^2}{r^2}\right) + V(r, \varphi),$$

and it was proven that only four independent potentials exist such that two quadratical integrals of motion exist.
In *G.Gubbiotti & MCN, J.Math.Phys., 2017* we have proven that the corresponding Lagrange equations can be linearized through hidden symmetries. N.B. Regardless of separability in the Hamilton-Jacobi equation, as shown in *MCN & S. Post, J.Phys.A, 2012* for the Post-Winternitz system *Post and Winternitz, J.Phys.A, 2011*.

Superintegrability and hidden linearity II

In *Friš, Mandrosov, Smorodinsky, Uhlìř and Winternitz, Phys.Lett. A, 1965* the following Hamiltonians were considered

$$H = \frac{1}{2}\left(p_1^2 + p_2^2\right) + V(x_1, x_2), \quad H = \frac{1}{2}\left(p_r^2 + \frac{p_\varphi^2}{r^2}\right) + V(r, \varphi),$$

and it was proven that only four independent potentials exist such that two quadratical integrals of motion exist.
In *G.Gubbiotti & MCN, J.Math.Phys., 2017* we have proven that the corresponding Lagrange equations can be linearized through hidden symmetries. N.B. Regardless of separability in the Hamilton-Jacobi equation, as shown in *MCN & S. Post, J.Phys.A, 2012* for the Post-Winternitz system *Post and Winternitz, J.Phys.A, 2011*. Among other cases, we have proven that the superintegrable Tremblay-Turbiner-Winternitz (TTW) equations hide linearizability through symmetries by raising the order.

Superintegrability and hidden linearity II

In *Friš, Mandrosov, Smorodinsky, Uhlìř and Winternitz, Phys.Lett. A, 1965* the following Hamiltonians were considered

$$H = \frac{1}{2}\left(p_1^2 + p_2^2\right) + V(x_1, x_2), \quad H = \frac{1}{2}\left(p_r^2 + \frac{p_\varphi^2}{r^2}\right) + V(r, \varphi),$$

and it was proven that only four independent potentials exist such that two quadratical integrals of motion exist.

In *G.Gubbiotti & MCN, J.Math.Phys., 2017* we have proven that the corresponding Lagrange equations can be linearized through hidden symmetries. N.B. Regardless of separability in the Hamilton-Jacobi equation, as shown in *MCN & S. Post, J.Phys.A, 2012* for the Post-Winternitz system *Post and Winternitz, J.Phys.A, 2011*. Among other cases, we have proven that the superintegrable Tremblay-Turbiner-Winternitz (TTW) equations hide linearizability through symmetries by raising the order.

Superintegrability and hidden linearity II

In *Friš, Mandrosov, Smorodinsky, Uhlìř and Winternitz, Phys.Lett. A, 1965* the following Hamiltonians were considered

$$H = \frac{1}{2}\left(p_1^2 + p_2^2\right) + V(x_1, x_2), \quad H = \frac{1}{2}\left(p_r^2 + \frac{p_\varphi^2}{r^2}\right) + V(r, \varphi),$$

and it was proven that only four independent potentials exist such that two quadratical integrals of motion exist.

In *G.Gubbiotti & MCN, J.Math.Phys., 2017* we have proven that the corresponding Lagrange equations can be linearized through hidden symmetries. N.B. Regardless of separability in the Hamilton-Jacobi equation, as shown in *MCN & S. Post, J.Phys.A,*

CONJECTURE:
ARE ALL SUPERINTEGRABLE SYSTEMS LINEARIZABLE IN 2-DIMENSIONAL SPACE?

hide linearizability through symmetries by raising the order.

TTW system: raising the order

The potential is

$$V(r,\varphi)=\omega^2r^2+\frac{k^2}{r^2}\left(\frac{\beta_1}{\cos^2(k\varphi)}+\frac{\beta_2}{\sin^2(k\varphi)}\right).$$

The Lagrangian equations are:

$$\ddot{r}=-4\omega^2r+r\dot{\varphi}^2+\frac{4k^2}{r^3}\left(\frac{\beta_1}{\cos^2(k\varphi)}+\frac{\beta_2}{\sin^2(k\varphi)}\right),$$

$$\ddot{\varphi}=-\frac{2\dot{r}\dot{\varphi}}{r}-\frac{4k^3}{r^4}\left(\frac{\beta_1\sin(k\varphi)}{\cos^3(k\varphi)}-\frac{\beta_2\cos(k\varphi)}{\sin^3(k\varphi)}\right).$$

They admit a three-dimensional Lie symmetry algebra generated by:

$$\Sigma_1=\partial_t,\quad \Sigma_2=\cos(4\omega t)\partial_t-2\omega\sin(4\omega t)r\partial_r,$$

$$\Sigma_3=\sin(4\omega t)\partial_t+2\omega\cos(4\omega t)r\partial_r,$$

By solving the Lagrangian equations with respect to β_1 and β_2, and then taking the derivative with respect to t, the following system of two equations of third order is obtained:

By solving the Lagrangian equations with respect to β_1 and β_2, and then taking the derivative with respect to t, the following system of two equations of third order is obtained:

$$r\dddot{r} + 16\omega^2 \dot{r} + 3\dot{r}\ddot{r} = 0, \quad (*)$$

$$\begin{aligned}
&\cos(k\varphi)\sin(k\varphi)r^2\dddot{\varphi} + 3\cos^2(k\varphi)kr^2\dot{\varphi}\ddot{\varphi} \\
&+ 6\cos^2(k\varphi)kr\dot{r}\dot{\varphi}^2 + 8\cos(k\varphi)\sin(k\varphi)\alpha k^2 r^2\dot{\varphi} \\
&- 4\cos(k\varphi)\sin(k\varphi)k^2r^2\dot{\varphi}^3 + 4\cos(k\varphi)\sin(k\varphi)k^2 r\ddot{r}\dot{\varphi} \\
&+ 6\cos(k\varphi)\sin(k\varphi)r\dot{r}\ddot{\varphi} + 2\cos(k\varphi)\sin(k\varphi)r\ddot{r}\dot{\varphi} \\
&+ 6\cos(k\varphi)\sin(k\varphi)\dot{r}^2\dot{\varphi} - 3\sin^2(k\varphi)kr^2\dot{\varphi}\ddot{\varphi} - 6\sin^2(k\varphi)kr\dot{r}\dot{\varphi}^2 = 0. \ (**)
\end{aligned}$$

By solving the Lagrangian equations with respect to β_1 and β_2, and then taking the derivative with respect to t, the following system of two equations of third order is obtained:

$$r\dddot{r} + 16\omega^2\dot{r} + 3\dot{r}\ddot{r} = 0, \quad (*)$$

$$\begin{aligned}
&\cos(k\varphi)\sin(k\varphi)r^2\dddot{\varphi} + 3\cos^2(k\varphi)kr^2\dot{\varphi}\ddot{\varphi}\\
&+ 6\cos^2(k\varphi)kr\dot{r}\dot{\varphi}^2 + 8\cos(k\varphi)\sin(k\varphi)\alpha k^2r^2\dot{\varphi}\\
&- 4\cos(k\varphi)\sin(k\varphi)k^2r^2\dot{\varphi}^3 + 4\cos(k\varphi)\sin(k\varphi)k^2r\ddot{r}\dot{\varphi}\\
&+ 6\cos(k\varphi)\sin(k\varphi)r\dot{r}\ddot{\varphi} + 2\cos(k\varphi)\sin(k\varphi)r\ddot{r}\dot{\varphi}\\
&+ 6\cos(k\varphi)\sin(k\varphi)\dot{r}^2\dot{\varphi} - 3\sin^2(k\varphi)kr^2\dot{\varphi}\ddot{\varphi} - 6\sin^2(k\varphi)kr\dot{r}\dot{\varphi}^2 = 0. \ (**)
\end{aligned}$$

The first equation admits a 7-dim Lie sym algebra generated by:
$X_1 = \partial_t,\ X_2 = \cos(4\omega t)\partial_t - 2\omega\sin(4\omega t)r\partial_r,$
$X_3 = \sin(4\omega t)\partial_t + 2\omega\cos(4\omega t)r\partial_r,\ X_4 = \frac{\cos(4\omega t)}{r}\partial_r,$
$X_5 = \frac{\sin(4\omega t)}{r}\partial_r,\ X_6 = r\partial_r,\ X_7 = \frac{1}{r}\partial_r,$ and consequently it is linearizable.

By solving the Lagrangian equations with respect to β_1 and β_2, and then taking the derivative with respect to t, the following system of two equations of third order is obtained:

$$r\dddot{r} + 16\omega^2\dot{r} + 3\dot{r}\ddot{r} = 0, \quad (*)$$

$$\begin{aligned}
&\cos(k\varphi)\sin(k\varphi)r^2\dddot{\varphi} + 3\cos^2(k\varphi)kr^2\dot{\varphi}\ddot{\varphi} \\
&+ 6\cos^2(k\varphi)kr\dot{r}\dot{\varphi}^2 + 8\cos(k\varphi)\sin(k\varphi)\alpha k^2 r^2\dot{\varphi} \\
&- 4\cos(k\varphi)\sin(k\varphi)k^2r^2\dot{\varphi}^3 + 4\cos(k\varphi)\sin(k\varphi)k^2 r\ddot{r}\dot{\varphi} \\
&+ 6\cos(k\varphi)\sin(k\varphi)r\dot{r}\ddot{\varphi} + 2\cos(k\varphi)\sin(k\varphi)r\ddot{r}\dot{\varphi} \\
&+ 6\cos(k\varphi)\sin(k\varphi)\dot{r}^2\dot{\varphi} - 3\sin^2(k\varphi)kr^2\dot{\varphi}\ddot{\varphi} - 6\sin^2(k\varphi)kr\dot{r}\dot{\varphi}^2 = 0. \ (**)
\end{aligned}$$

The first equation admits a 7-dim Lie sym algebra generated by:
$X_1 = \partial_t,\ X_2 = \cos(4\omega t)\partial_t - 2\omega\sin(4\omega t)r\partial_r,$
$X_3 = \sin(4\omega t)\partial_t + 2\omega\cos(4\omega t)r\partial_r,\ X_4 = \frac{\cos(4\omega t)}{r}\partial_r,$
$X_5 = \frac{\sin(4\omega t)}{r}\partial_r,\ X_6 = r\partial_r,\ X_7 = \frac{1}{r}\partial_r,$ and consequently it is linearizable. A 2-dim nonabelian intransitive subalgebra is $< X_6, X_7 >$, that yields $u = r^2/2$ and thus $\dddot{u} = -16\omega^2\dot{u}$. Finally $r = \sqrt{a_1 + a_2\cos(4\omega t) + a_3\sin(4\omega t)}$.

Also the second equation (**) is linearizable since it admits a 7-dim Lie symmetry algebra generated by:

$$\Omega = s_1(t)\partial_t + \frac{-\cos^2(k\varphi)s_2(t) + 2ks_3(t)}{2\cos(k\varphi)\sin(k\varphi)k}\partial_\varphi,$$

with s_1, s_2, s_3 that satisfy the following seventh-order linear system:

$$\begin{aligned}
&r^2\dddot{s}_1 + 4\dot{s}_1\ddot{r}k^2r - 4\dot{s}_1\ddot{r}r + 16\dot{s}_1k^2\omega^2r^2 - 8\ddot{r}\dot{r}k^2s_1 \\
&\qquad + 8\ddot{r}\dot{r}s_1 - 32\dot{r}k^2\omega^2s_1r + 32\dot{r}\omega^2s_1r = 0, \\
&r^2\dot{s}_2 - \ddot{s}_1r^2 + 2\dot{s}_1r\dot{r} + 2r\ddot{r}s_1 - 2\dot{r}^2s_1 = 0, \\
&r^2\dddot{s}_3 + 6\ddot{s}_3\dot{r}r + 4\dot{s}_3\ddot{r}k^2r + \dot{s}_3\ddot{r}r + 6\dot{s}_3\dot{r}^2 + 16\dot{s}_3k^2\omega^2r^2 = 0.
\end{aligned}$$

Also the second equation (**) is linearizable since it admits a 7-dim Lie symmetry algebra generated by:

$$\Omega = s_1(t)\partial_t + \frac{-\cos^2(k\varphi)s_2(t) + 2ks_3(t)}{2\cos(k\varphi)\sin(k\varphi)k}\partial_\varphi,$$

with s_1, s_2, s_3 that satisfy the following seventh-order linear system:

$$\begin{aligned} & r^2 \dddot{s}_1 + 4\dot{s}_1\ddot{r}k^2 r - 4\dot{s}_1\ddot{r}r + 16\dot{s}_1 k^2\omega^2 r^2 - 8\ddot{r}\dot{r}k^2 s_1 \\ & \qquad + 8\ddot{r}\dot{r}s_1 - 32\dot{r}k^2\omega^2 s_1 r + 32\dot{r}\omega^2 s_1 r = 0, \\ & r^2\dot{s}_2 - \ddot{s}_1 r^2 + 2\dot{s}_1 r\dot{r} + 2r\ddot{r}s_1 - 2\dot{r}^2 s_1 = 0, \\ & r^2 \dddot{s}_3 + 6\ddot{s}_3\dot{r}r + 4\dot{s}_3\ddot{r}k^2 r + \dot{s}_3\ddot{r}r + 6\dot{s}_3\dot{r}^2 + 16\dot{s}_3 k^2\omega^2 r^2 = 0. \end{aligned}$$

Also the second equation (**) is linearizable since it admits a 7-dim Lie symmetry algebra generated by:

$$\Omega = s_1(t)\partial_t + \frac{-\cos^2(k\varphi)s_2(t) + 2ks_3(t)}{2\cos(k\varphi)\sin(k\varphi)k}\partial_\varphi,$$

with s_1, s_2, s_3 that satisfy the following seventh-order linear system:

$$\begin{aligned}
&r^2 \dddot{s}_1 + 4\dot{s}_1\ddot{r}k^2 r - 4\dot{s}_1\ddot{r}r + 16\dot{s}_1 k^2\omega^2 r^2 - 8\ddot{r}\dot{r}k^2 s_1 \\
&\qquad\qquad + 8\ddot{r}\dot{r}s_1 - 32\dot{r}k^2\omega^2 s_1 r + 32\dot{r}\omega^2 s_1 r = 0, \\
&r^2\dot{s}_2 - \ddot{s}_1 r^2 + 2\dot{s}_1 r\dot{r} + 2r\ddot{r}s_1 - 2\dot{r}^2 s_1 = 0, \\
&r^2 \dddot{s}_3 + 6\ddot{s}_3\dot{r}r + 4\dot{s}_3\ddot{r}k^2 r + \dot{s}_3\ddot{r}r + 6\dot{s}_3\dot{r}^2 + 16\dot{s}_3 k^2\omega^2 r^2 = 0.
\end{aligned}$$

Also the second equation (**) is linearizable since it admits a 7-dim Lie symmetry algebra generated by:

$$\Omega = s_1(t)\partial_t + \frac{-\cos^2(k\varphi)s_2(t) + 2ks_3(t)}{2\cos(k\varphi)\sin(k\varphi)k}\partial_\varphi,$$

with s_1, s_2, s_3 that satisfy the following seventh-order linear system:

$$\begin{aligned}
&r^2\dddot{s}_1 + 4\dot{s}_1\ddot{r}k^2 r - 4\dot{s}_1\ddot{r}r + 16\dot{s}_1 k^2\omega^2 r^2 - 8\ddot{r}\dot{r}k^2 s_1 \\
&\qquad + 8\ddot{r}\dot{r}s_1 - 32\dot{r}k^2\omega^2 s_1 r + 32\dot{r}\omega^2 s_1 r = 0, \\
&r^2\dot{s}_2 - \ddot{s}_1 r^2 + 2\dot{s}_1 r\dot{r} + 2r\ddot{r}s_1 - 2\dot{r}^2 s_1 = 0, \\
&r^2\dddot{s}_3 + 6\ddot{s}_3\dot{r}r + 4\dot{s}_3\ddot{r}k^2 r + \dot{s}_3\ddot{r}r + 6\dot{s}_3\dot{r}^2 + 16\dot{s}_3 k^2\omega^2 r^2 = 0.
\end{aligned}$$

$< -\frac{1}{2k}\cot(k\varphi)\partial_\varphi, \frac{2}{\sin(2k\varphi)}\partial_\varphi >$ is a 2-dim nonabelian intransitive subalgebra. Then the second equation (**) becomes linear by means of the canonical transformation $v = -\frac{1}{2k}\cos^2(k\varphi)$, i.e.

$$\dddot{v} = -\frac{6\dot{r}}{r}\ddot{v} - \frac{2}{r^2}\left(3\dot{r}^2 + 8k^2\omega^2 r^2 + (2k^2+1)r\ddot{r}\right)\dot{v}.$$

More hidden linearity

In *G.Gubbiotti & MCN, 2019* we have determined the hidden linearity of Darboux space of Type I&II *Tremblay, Turbiner, Winternitz, J.Math.Phys., 2002*, and Bertrand-Perlick *Perlick, Class. Quant. Grav., 1992* (i.e., Type III&IV, see *Riglioni, J.Phys.A, 2013*).

More hidden linearity

In *G.Gubbiotti & MCN, 2019* we have determined the hidden linearity of Darboux space of Type I&II *Tremblay, Turbiner, Winternitz, J.Math.Phys., 2002*, and Bertrand-Perlick *Perlick, Class. Quant. Grav., 1992* (i.e., Type III&IV, see *Riglioni, J.Phys.A, 2013*). In *Evans, Phys.Rev. A, 1990*, four Hamiltonians were derived. From *MCN & R.Campoamor-Stursberg, 2019* let us show that H_3 hides linearity **N.B. Hamilton-Jacobi eq is NOT separable in cartesian coordinates**.

More hidden linearity

In *G.Gubbiotti & MCN, 2019* we have determined the hidden linearity of Darboux space of Type I&II *Tremblay, Turbiner, Winternitz, J.Math.Phys., 2002*, and Bertrand-Perlick *Perlick, Class. Quant. Grav., 1992* (i.e., Type III&IV, see *Riglioni, J.Phys.A, 2013*). In *Evans, Phys.Rev. A, 1990*, four Hamiltonians were derived. From *MCN & R.Campoamor-Stursberg, 2019* let us show that H_3 hides linearity **N.B. Hamilton-Jacobi eq is NOT separable in cartesian coordinates**.

$$H_3 = \frac{1}{2}\left(p_1^2 + p_2^2 + p_3^2\right) + \frac{k_1 w_1}{w_2^2\sqrt{w_1^2 + w_2^2}} + \frac{k_2}{w_2^2} + \frac{k_3}{w_3^2}$$

yields the Hamilton equations

$$\dot{w}_1 = p_1, \quad \dot{w}_2 = p_2, \quad \dot{w}_3 = p_3, \quad \dot{p}_1 = -\frac{k_1}{(w_1^2 + w_2^2)^{3/2}},$$

$$\dot{p}_2 = \frac{k_1 w_1(2w_1^2 + 3w_2^2)}{w_2^3(w_1^2 + w_2^2)^{3/2}} + \frac{2k_2}{w_2^3}, \quad \dot{p}_3 = \frac{2k_3}{w_3^3}.$$

More hidden linearity

In *G.Gubbiotti & MCN, 2019* we have determined the hidden linearity of Darboux space of Type I&II *Tremblay, Turbiner, Winternitz, J.Math.Phys., 2002*, and Bertrand-Perlick *Perlick, Class. Quant. Grav., 1992* (i.e., Type III&IV, see *Riglioni, J.Phys.A, 2013*). In *Evans, Phys.Rev. A, 1990*, four Hamiltonians were derived. From *MCN & R.Campoamor-Stursberg, 2019* let us show that H_3 hides linearity **N.B. Hamilton-Jacobi eq is NOT separable in cartesian coordinates**.

$$H_3 = \frac{1}{2}\left(p_1^2 + p_2^2 + p_3^2\right) + \frac{k_1 w_1}{w_2^2\sqrt{w_1^2 + w_2^2}} + \frac{k_2}{w_2^2} + \frac{k_3}{w_3^2}$$

yields the Hamilton equations

$$\dot{w}_1 = p_1, \quad \dot{w}_2 = p_2, \quad \dot{w}_3 = p_3, \quad \dot{p}_1 = -\frac{k_1}{(w_1^2 + w_2^2)^{3/2}},$$

$$\dot{p}_2 = \frac{k_1 w_1 (2w_1^2 + 3w_2^2)}{w_2^3 (w_1^2 + w_2^2)^{3/2}} + \frac{2k_2}{w_2^3}, \quad \dot{p}_3 = \frac{2k_3}{w_3^3}.$$

Then $p_3 = \dot{w}_3 \quad \Longrightarrow \quad \ddot{w}_3 = \dfrac{2k_3}{w_3^3},$

Solving $\ddot{w}_3 = \dfrac{2k_3}{w_3^3}$ with respect to k_3 and deriving once with respect to t, yields:

$$\dddot{w}_3 = -\frac{3\dot{w}_3\ddot{w}_3}{w_3},$$

which admits a 7-dim Lie symmetry algebra generated by

$$X_1 = t^2\partial_t + tw_3\partial_{w_3},\ X_2 = t\partial_t,\ X_3 = \partial_t,\ X_4 = w_3\partial_{w_3},$$
$$X_5 = \frac{t^2}{w_3}\partial_{w_3},\ X_6 = \frac{t}{w_3}\partial_{w_3},\ X_7 = \frac{1}{w_3}\partial_{w_3},$$

and therefore it is linearizable. In fact, the new dependent variable $u = w_3^2/2$ transforms it into the linear equation

$$\dddot{u} = 0,$$

and thus the general solution is

$$w_3 = \pm\sqrt{A_1 t^2 + A_2 t + \frac{A_2^2 + 8k_3}{4A_1}},$$

with $A_n, (n = 1, 2)$ arbitrary constants.

About the other four equations of the Hamilton H_3 system:

$$\dot{w}_1 = p_1, \quad \dot{w}_2 = p_2, \quad \dot{p}_1 = -\frac{k_1}{(w_1^2 + w_2^2)^{3/2}},$$

$$\dot{p}_2 = \frac{k_1 w_1 (2w_1^2 + 3w_2^2)}{w_2^3 (w_1^2 + w_2^2)^{3/2}} + \frac{2k_2}{w_2^3},$$

we make the simplifying substitution $w_2 = \sqrt{r_2^2 - w_1^2}$.

Then, deriving p_1 we obtain,

$$\ddot{w}_1 = -\frac{k_1}{r_2^3}.$$

Deriving p_2 we obtain:

$$\ddot{r}_2 = \frac{w_1^2 \dot{r}_2^2}{r_2(r_2^2 - w_1^2)} - \frac{2w_1 \dot{w}_1 \dot{r}_2}{r_2^2 - w_1^2} + \frac{\dot{w}_1^2 r_2^3 + 2k_1 w_1 + 2k_2 r_2}{r_2^2(r_2^2 - w_1^2)}.$$

The system w_1, r_2 admits a 3-dim Lie symmetry algebra $sl(2, \mathbb{R})$ generated by:

$$t^2 \partial_t + t w_1 \partial_{w_1} + t r_2 \partial_{r_2}, \quad 2t\partial_t + w_1 \partial_{w_1} + r_2 \partial_{r_2}, \quad \partial_t.$$

If we solve system

$$
\begin{aligned}
\ddot{w}_1 &= -\frac{k_1}{r_2^3}, \\
\ddot{r}_2 &= \frac{w_1^2 \dot{r}_2^2}{r_2(r2^2 - w_1^2)} - \frac{2 w_1 \dot{w}_1 \dot{r}_2}{r2^2 - w_1^2} + \frac{\dot{w}_1^2 r_2^3 + 2 k_1 w_1 + 2 k_2 r_2}{r_2^2(r2^2 - w_1^2)},
\end{aligned}
$$

with respect to k_1, k_2 and derive once, we obtain:

$$
\dddot{w}_1 = -\frac{3 \dot{w}_1 \ddot{w}_1}{w_1}, \qquad \dddot{r}_2 = -\frac{3 \dot{r}_2 \ddot{r}_2}{r_2},
$$

namely both w_1 and r_2 satisfy the same equation as w_3.

If we solve system

$$
\begin{aligned}
\ddot{w}_1 &= -\frac{k_1}{r_2^3}, \\
\ddot{r}_2 &= \frac{w_1^2 \dot{r}_2^2}{r_2(r2^2 - w_1^2)} - \frac{2 w_1 \dot{w}_1 \dot{r}_2}{r2^2 - w_1^2} + \frac{\dot{w}_1^2 r_2^3 + 2k_1 w_1 + 2k_2 r_2}{r_2^2(r2^2 - w_1^2)},
\end{aligned}
$$

with respect to k_1, k_2 and derive once, we obtain:

$$\dddot{w}_1 = -\frac{3\dot{w}_1 \ddot{w}_1}{w_1}, \qquad \dddot{r}_2 = -\frac{3\dot{r}_2 \ddot{r}_2}{r_2},$$

namely both w_1 and r_2 satisfy the same equation as w_3.
Consequently, the transformations $u_1 = w_1^2/2, u_2 = r_2^2/2$ yield:

$$\dddot{u}_1 = 0, \qquad \dddot{u}_2 = 0.$$

If we solve system

$$\begin{aligned}\ddot{w}_1 &= -\frac{k_1}{r_2^3},\\ \ddot{r}_2 &= \frac{w_1^2 \dot{r}_2^2}{r_2(r2^2 - w_1^2)} - \frac{2 w_1 \dot{w}_1 \dot{r}_2}{r2^2 - w_1^2} + \frac{\dot{w}_1^2 r_2^3 + 2k_1 w_1 + 2k_2 r_2}{r_2^2(r2^2 - w_1^2)},\end{aligned}$$

with respect to k_1, k_2 and derive once, we obtain:

$$\dddot{w}_1 = -\frac{3\dot{w}_1 \ddot{w}_1}{w_1}, \qquad \dddot{r}_2 = -\frac{3\dot{r}_2 \ddot{r}_2}{r_2},$$

namely both w_1 and r_2 satisfy the same equation as w_3. Consequently, the transformations $u_1 = w_1^2/2, u_2 = r_2^2/2$ yield:

$$\dddot{u}_1 = 0, \qquad \dddot{u}_2 = 0.$$

Indeed, the Hamiltonian system H_3 hides (three times) the linear equation $\dddot{u} = 0$.

If we solve system

$$\ddot{w}_1 = -\frac{k_1}{r_2^3},$$

$$\ddot{r}_2 = \frac{w_1^2 \dot{r}_2^2}{r_2(r2^2 - w_1^2)} - \frac{2 w_1 \dot{w}_1 \dot{r}_2}{r2^2 - w_1^2} + \frac{\dot{w}_1^2 r_2^3 + 2k_1 w_1 + 2k_2 r_2}{r_2^2(r2^2 - w_1^2)},$$

with respect to k_1, k_2 and derive once, we obtain:

$$\dddot{w}_1 = -\frac{3\dot{w}_1 \ddot{w}_1}{[illegible]} \qquad \dddot{r}_2 = -\frac{3\dot{r}_2 \ddot{r}_2}{[illegible]}$$

CONJECTURE:
ARE ALL MAXIMALLY SUPERINTEGRABLE SYSTEMS IN 3-DIM LINEARIZABLE?

$$\dddot{u}_1 = 0, \qquad \dddot{u}_2 = 0.$$

Indeed, the Hamiltonian system H_3 hides (three times) the linear equation $\dddot{u} = 0$.

Ten Lagrangians

Ten different JLM and consequently as many Lagrangians:

$$M_{13} = -\frac{1}{(t\dot{x}-x)^3} \quad \Rightarrow \quad L_{1,3} = -\frac{1}{2t^2(t\dot{x}-x)} + \frac{\mathrm{d}g}{\mathrm{d}t}(t,x)$$

$$M_{15} = -\frac{1}{\dot{x}(t\dot{x}-x)^2} \quad \Rightarrow \quad L_{1,5} = \frac{\dot{x}}{x^2}\left(\log(t\dot{x}-x) - \log(\dot{x})\right)$$

$$M_{16} = \frac{1}{\dot{x}^2(t\dot{x}-x)} \quad \Rightarrow \quad L_{1,6} = \left(\frac{t\dot{x}}{x^2} - \frac{1}{x}\right)\left(\log(\dot{x}) - \log(t\dot{x}-x)\right)$$

$$M_{17} = -\frac{1}{(t\dot{x}-x)^2} \quad \Rightarrow \quad L_{1,7} = -\frac{1}{t^2}\log(t\dot{x}-x)$$

$$M_{18} = \frac{1}{\dot{x}(t\dot{x}-x)} \quad \Rightarrow \quad L_{1,8} = -\frac{\dot{x}}{x}\log(\dot{x}) - \left(\frac{1}{t} - \frac{\dot{x}}{x}\right)\log(t\dot{x}-x) + \frac{1}{t}(1+\log(x))$$

$$M_{62} = \frac{1}{\dot{x}^3} \quad \Rightarrow \quad L_{6,2} = \frac{1}{2\dot{x}}$$

$$M_{28} = \frac{1}{\dot{x}^2} \quad \Rightarrow \quad L_{2,8} = -\log(\dot{x})$$

$$M_{38} = \frac{1}{t\dot{x} - x} \quad \Rightarrow \quad L_{3,8} = \left(\frac{\dot{x}}{t} - \frac{x}{t^2}\right)(\log(t\dot{x} - x) - 1)$$

$$M_{48} = -\frac{1}{\dot{x}} \quad \Rightarrow \quad L_{4,8} = \dot{x}(1 - \log(\dot{x}))$$

$$M_{87} = 1 \quad \Rightarrow$$

$$M_{62} = \frac{1}{\dot{x}^3} \quad \Rightarrow \quad L_{6,2} = \frac{1}{2\dot{x}}$$

$$M_{28} = \frac{1}{\dot{x}^2} \quad \Rightarrow \quad L_{2,8} = -\log(\dot{x})$$

$$M_{38} = \frac{1}{t\dot{x} - x} \quad \Rightarrow \quad L_{3,8} = \left(\frac{\dot{x}}{t} - \frac{x}{t^2}\right)(\log(t\dot{x} - x) - 1)$$

$$M_{48} = -\frac{1}{\dot{x}} \quad \Rightarrow \quad L_{4,8} = \dot{x}(1 - \log(\dot{x}))$$

$$M_{87} = 1 \quad \Rightarrow \quad L_{8,7} = \tfrac{1}{2}\dot{x}^2$$

FINALLY, THE TRUE LAGRANGIAN

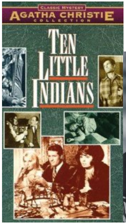
How do we (physically) eliminate 9 out of 10??

How do we (physically) eliminate 9 out of 10??

CALL ....AGATHA CHRISTIE??

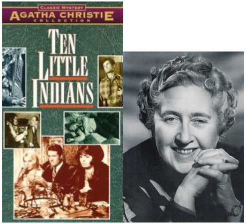
How do we (physically) eliminate 9 out of 10??

CALL ……..AGATHA CHRISTIE??



How do we (physically) eliminate 9 out of 10??

CALL ....AGATHA CHRISTIE??



NO!!! CALL...

How do we (physically) eliminate 9 out of 10??

CALL ....

N

How do we (physically) eliminate 9 out of 10??

Which Quantization?

Different methods exist to describe the dynamics of quantum systems, among them:

Which Quantization?

Different methods exist to describe the dynamics of quantum systems, among them:

(i) the time-dependent Schrödinger equation with the corresponding continuity equation for the probability density;

Which Quantization?

Different methods exist to describe the dynamics of quantum systems, among them:

(i) the time-dependent Schrödinger equation with the corresponding continuity equation for the probability density;

(ii) the description using a time propagator, also called Feynman kernel or space representation of the Green function;

Which Quantization?

Different methods exist to describe the dynamics of quantum systems, among them:

(i) the time-dependent Schrödinger equation with the corresponding continuity equation for the probability density;

(ii) the description using a time propagator, also called Feynman kernel or space representation of the Green function;

(iii) the time-dependent Wigner function.

Which Quantization?

Different methods exist to describe the dynamics of quantum systems, among them:

(i) the time-dependent Schrödinger equation with the corresponding continuity equation for the probability density;

(ii) the description using a time propagator, also called Feynman kernel or space representation of the Green function;

(iii) the time-dependent Wigner function.

All three methods were shown to be linked by the Ermakov invariant in *D. Schuch & M. Moshinsky, Phys.Rev.A, 2006*.

Which Quantization?

Different methods exist to describe the dynamics of quantum systems, among them:

- **(i)** the time-dependent Schrödinger equation with the corresponding continuity equation for the probability density;
- **(ii)** the description using a time propagator, also called Feynman kernel or space representation of the Green function;
- **(iii)** the time-dependent Wigner function.

All three methods were shown to be linked by the Ermakov invariant in *D. Schuch & M. Moshinsky, Phys.Rev.A, 2006*. Therefore we pursue the quantization of classical problems by searching for a time-dependent Schrödinger equation.

How to obtain the Schrödinger equation from Noether symmetries I

$$\ddot{q} = 0$$

$L = \frac{1}{2}\dot{q}^2$ admits five Noether symmetries:

$$X_1 = \partial_t,\ X_2 = \partial_q,\ X_3 = t\partial_q,\ X_4 = 2t\partial_t + q\partial_q,\ X_5 = t^2\partial_t + tq\partial_q.$$

How to obtain the Schrödinger equation from Noether symmetries I

$$\ddot{q} = 0$$

$L = \frac{1}{2}\dot{q}^2$ admits five Noether symmetries:

$$X_1 = \partial_t,\ X_2 = \partial_q,\ X_3 = t\partial_q,\ X_4 = 2t\partial_t + q\partial_q,\ X_5 = t^2\partial_t + tq\partial_q.$$

The Schrödinger equation

$$2i\psi_t + \psi_{qq} = 0$$

admits $5 + 1 + \infty$ Lie symmetries:

$$Y_1 = X_1,\ Y_2 = X_2,\ Y_3 = X_3 + iq\psi\partial_\psi,\ Y_4 = X_4,$$
$$Y_5 = X_5 + \frac{1}{2}(iq^2 - t)\psi\partial_\psi.$$

plus $\psi\partial_\psi$ and $\alpha(t,q)\partial_\psi$ s.t. $2i\alpha_t + \alpha_{qq} = 0$

Marcos Moshinsky

He was used to say [K.B. Wolf, J. Phys.: Conf. Ser. 237 (2010)]:

Marcos Moshinsky

He was used to say [K.B. Wolf, J. Phys.: Conf. Ser. 237 (2010)]:

Two types of problems exist in quantum mechanics, those that you cannot solve and the harmonic oscillator. The trick is to push a problem from one category to the other.

How to obtain the Schrödinger equation from Noether symmetries II

$$\ddot{x} = -\omega^2 x$$

How to obtain the Schrödinger equation from Noether symmetries II

$$\ddot{x} = -\omega^2 x$$

$L = \frac{1}{2}(\dot{x}^2 - \omega^2 x^2)$ admits 5 Noether symmetries:

$$\begin{aligned} X_1 &= \partial_t,\ X_2 = \cos(2\omega t)\partial_t - \omega x \sin(2\omega t)\partial_x, \\ X_3 &= \sin(2\omega t)\partial_t + \omega x \cos(2\omega t)\partial_x,\ X_4 = \cos(\omega t)\partial_x, \\ X_5 &= \sin(\omega t)\partial_x. \end{aligned}$$

How to obtain the Schrödinger equation from Noether symmetries II

$$\ddot{x} = -\omega^2 x$$

$L = \frac{1}{2}(\dot{x}^2 - \omega^2 x^2)$ admits 5 Noether symmetries:

$$\begin{aligned}
X_1 &= \partial_t,\ X_2 = \cos(2\omega t)\partial_t - \omega x \sin(2\omega t)\partial_x, \\
X_3 &= \sin(2\omega t)\partial_t + \omega x \cos(2\omega t)\partial_x,\ X_4 = \cos(\omega t)\partial_x, \\
X_5 &= \sin(\omega t)\partial_x.
\end{aligned}$$

The Schrödinger equation (with $\hbar = 1$)

$$2i\psi_t + \psi_{xx} - \omega^2 x^2 \psi = 0$$

How to obtain the Schrödinger equation from Noether symmetries II

$$\ddot{x} = -\omega^2 x$$

$L = \frac{1}{2}(\dot{x}^2 - \omega^2 x^2)$ admits 5 Noether symmetries:

$$\begin{aligned} X_1 &= \partial_t,\ X_2 = \cos(2\omega t)\partial_t - \omega x \sin(2\omega t)\partial_x, \\ X_3 &= \sin(2\omega t)\partial_t + \omega x \cos(2\omega t)\partial_x,\ X_4 = \cos(\omega t)\partial_x, \\ X_5 &= \sin(\omega t)\partial_x. \end{aligned}$$

The Schrödinger equation (with $\hbar = 1$)

$$2i\psi_t + \psi_{xx} - \omega^2 x^2 \psi = 0$$

admits the $5 + 1 + \infty$ dimensional Lie symmetry algebra:

$$\begin{aligned} Y_1 &= X_1,\ Y_2 = X_2 + \omega(\sin(2\omega t) - 2i\omega x^2 \cos(2\omega t))\psi\partial_\psi, \\ Y_3 &= X_3 - \omega(\cos(2\omega t) - 2i\omega x^2 \sin(2\omega t))\psi\partial_\psi, \\ Y_4 &= X_4 - 2i\omega x \sin(\omega t)\psi\partial_\psi,\ Y_5 = X_5 + 2i\omega x \cos(\omega t)\psi\partial_\psi. \end{aligned}$$

plus $\psi\partial_\psi$ and $\alpha(t,x)\partial_\psi$ s.t. $2i\alpha_t + \alpha_{xx} - \omega^2 x^2 \alpha = 0$

Quantizing with Noether: Act 1

MCN, Theor.Math.Phys., 2011

Quantizing with Noether: Act 1

MCN, Theor.Math.Phys., 2011

- Find the Lie symmetries of the Lagrange equation

$$\Upsilon = W_0(t,x)\partial_t + W_1(t,x)\partial_x$$

Quantizing with Noether: Act 1

MCN, Theor.Math.Phys., 2011

- Find the Lie symmetries of the Lagrange equation

$$\Upsilon = W_0(t,x)\partial_t + W_1(t,x)\partial_x$$

- Find the Noether symmetries

$$\Gamma = V_0(t,x)\partial_t + V_1(t,x)\partial_x, \qquad \Gamma \in \Upsilon$$

Quantizing with Noether: Act 1

MCN, Theor.Math.Phys., 2011

- Find the Lie symmetries of the Lagrange equation

$$\Upsilon = W_0(t,x)\partial_t + W_1(t,x)\partial_x$$

- Find the Noether symmetries

$$\Gamma = V_0(t,x)\partial_t + V_1(t,x)\partial_x, \qquad \Gamma \in \Upsilon$$

- Construct the Schrödinger equation admitting those Lie symmetries

$$2i\psi_t + f_1(x)\psi_{xx} + f_2(x)\psi_x + f_3(x)\psi = 0$$

$$\Omega = V_0(t,x)\partial_t + V_1(t,x)\partial_x + G(t,x,\psi)\partial_\psi$$

Quantizing with Noether: Act 1

MCN, Theor.Math.Phys., 2011

- Find the Lie symmetries of the Lagrange equation

$$\Upsilon = W_0(t,x)\partial_t + W_1(t,x)\partial_x$$

- Find the Noether symmetries

$$\Gamma = V_0(t,x)\partial_t + V_1(t,x)\partial_x, \qquad \Gamma \in \Upsilon$$

- Construct the Schrödinger equation admitting those Lie symmetries

$$2i\psi_t + f_1(x)\psi_{xx} + f_2(x)\psi_x + f_3(x)\psi = 0$$

$$\Omega = V_0(t,x)\partial_t + V_1(t,x)\partial_x + G(t,x,\psi)\partial_\psi$$

- N.B. $\psi\partial_\psi$ and $\Psi(t,x)\partial_\psi$

Quantizing with Noether: Act 1

MCN, Theor.Math.Phys., 2011

- Find the Lie symmetries of the Lagrange equation

$$\Upsilon = W_0(t,x)\partial_t + W_1(t,x)\partial_x$$

- Find the Noether symmetries

$$\Gamma = V_0(t,x)\partial_t + V_1(t,x)\partial_x, \qquad \Gamma \in \Upsilon$$

- Construct the Schrödinger equation admitting those Lie symmetries

$$2i\psi_t + f_1(x)\psi_{xx} + f_2(x)\psi_x + f_3(x)\psi = 0$$

$$\Omega = V_0(t,x)\partial_t + V_1(t,x)\partial_x + G(t,x,\psi)\partial_\psi$$

QUANTIZE PRESERVING THE SYMMETRIES!

An example from population dynamics

MCN & G. Sanchini, Symmetry, 2016 Equation

$$\ddot{u} = \frac{5\dot{u}^2}{4u} - \frac{2c^2}{K}u^2 - c^2 u$$

admits a 3-dim Lie (**complete**) symmetry group: ★★★

$$\Gamma_1 = \partial_t, \qquad \Gamma_2 = e^{-ct}(\partial_t + 2cu\partial_u) \qquad \Gamma_3 = e^{ct}(\partial_t - 2cu\partial_u).$$

An example from population dynamics

MCN & G. Sanchini, Symmetry, 2016 Equation

$$\ddot{u} = \frac{5\dot{u}^2}{4u} - \frac{2c^2}{K}u^2 - c^2 u$$

admits a 3-dim Lie (**complete**) symmetry group: ★★★

$$\Gamma_1 = \partial_t, \qquad \Gamma_2 = e^{-ct}(\partial_t + 2cu\partial_u) \qquad \Gamma_3 = e^{ct}(\partial_t - 2cu\partial_u).$$

The Lagrangian with those 3 Noether symmetries is

$$L = \sqrt{u}\left(\frac{\dot{u}^2}{4cu^3} + \frac{c}{u} - \frac{2c}{K}\right)$$

An example from population dynamics

MCN & G. Sanchini, Symmetry, 2016 Equation

$$\ddot{u} = \frac{5\dot{u}^2}{4u} - \frac{2c^2}{K}u^2 - c^2 u$$

admits a 3-dim Lie (**complete**) symmetry group: ★★★

$$\Gamma_1 = \partial_t, \qquad \Gamma_2 = e^{-ct}(\partial_t + 2cu\partial_u) \qquad \Gamma_3 = e^{ct}(\partial_t - 2cu\partial_u).$$

The Lagrangian with those 3 Noether symmetries is

$$L = \sqrt{u}\left(\frac{\dot{u}^2}{4cu^3} + \frac{c}{u} - \frac{2c}{K}\right)$$

and the Schrödinger equation is

$$2\mathrm{i}\psi_t + u^2\sqrt{u}\psi_{uu} - \sqrt{u}\left(\eta + 4\frac{c^2}{u}\right)\psi = 0,$$

An example from population dynamics

MCN & G. Sanchini, Symmetry, 2016 Equation

$$\ddot{u} = \frac{5\dot{u}^2}{4u} - \frac{2c^2}{K}u^2 - c^2 u$$

admits a 3-dim Lie (**complete**) symmetry group: ★★★

$$\Gamma_1 = \partial_t, \qquad \Gamma_2 = e^{-ct}\left(\partial_t + 2cu\partial_u\right) \qquad \Gamma_3 = e^{ct}\left(\partial_t - 2cu\partial_u\right).$$

The Lagrangian with those 3 Noether symmetries is

$$L = \sqrt{u}\left(\frac{\dot{u}^2}{4cu^3} + \frac{c}{u} - \frac{2c}{K}\right)$$

and the Schrödinger equation is

$$2\mathrm{i}\psi_t + u^2\sqrt{u}\psi_{uu} - \sqrt{u}\left(\eta + 4\frac{c^2}{u}\right)\psi = 0,$$

$$\text{with}: \qquad \Lambda_1 = \Gamma_1, \qquad \Lambda_2 = \Gamma_2 + ce^{-ct}\left(\frac{3}{2} + 4\mathrm{i}\frac{c}{\sqrt{u}}\right)\psi\partial_\psi,$$

$$\Lambda_3 = \Gamma_3 + ce^{ct}\left(-\frac{3}{2} + 4\mathrm{i}\frac{c}{\sqrt{u}}\right)\psi\partial_\psi.$$

Charged particle in a uniform magnetic field

Its classical Lagrangian is

$$L = \frac{1}{2}\left((\dot{x}^2 + \dot{y}^2) + \omega(y\dot{x} - x\dot{y})\right) \qquad (5)$$

and consequently the Lagrangian equations are

$$\ddot{x} = -\omega\dot{y}, \quad \ddot{y} = \omega\dot{x}.$$

Charged particle in a uniform magnetic field

Its classical Lagrangian is

$$L = \frac{1}{2}\left((\dot{x}^2 + \dot{y}^2) + \omega(y\dot{x} - x\dot{y})\right) \quad (5)$$

and consequently the Lagrangian equations are

$$\ddot{x} = -\omega\dot{y}, \quad \ddot{y} = \omega\dot{x}.$$

The Lagrangian (5) admits 8 Noether symmetries generated by

$$\begin{aligned}
X_1 &= \cos(\omega t)\partial_t - \frac{\omega}{2}\left(\sin(\omega t)x + \cos(\omega t)y\right)\partial_x \\
&\quad + \frac{\omega}{2}\left(\cos(\omega t)x - \sin(\omega t)y\right)\partial_y, \\
X_2 &= -\sin(\omega t)\partial_t - \frac{1}{2}\left(\cos(\omega t)\omega x - \sin(\omega t)\omega y\right)\partial_x \\
&\quad - \frac{1}{2}(\sin(\omega t)\omega x + \cos(\omega t)\omega y)\partial_y, \\
X_3 &= \partial_t, \quad X_4 = -y\partial_x + x\partial_y, \quad X_5 = -\sin(\omega t)\partial_x + \cos(\omega t)\partial_y, \\
X_6 &= -\cos(\omega t)\partial_x - \sin(\omega t)\partial_y, \quad X_7 = \partial_y, \quad X_8 = \partial_x.
\end{aligned}$$

The Schrödinger equation was determined by Sir Charles Galton Darwin, Proc. R. Soc. Lond. A, 1927 to be

$$2i\psi_t + \psi_{xx} + \psi_{yy} - i\omega(y\psi_x - x\psi_y) - \frac{\omega^2}{4}(x^2 + y^2)\psi = 0. \qquad (6)$$

The Schrödinger equation was determined by Sir Charles Galton Darwin, Proc. R. Soc. Lond. A, 1927 to be

$$2i\psi_t + \psi_{xx} + \psi_{yy} - i\omega(y\psi_x - x\psi_y) - \frac{\omega^2}{4}(x^2 + y^2)\psi = 0. \qquad (6)$$

It admits the $8 + 1 + \infty$ dim Lie symmetry algebra, i.e.:

$$\begin{aligned}
Y_1 &= X_1 + \frac{1}{4}\left(2\sin(\omega t)\omega - i\cos(\omega t)\omega^2(x^2+y^2)\right)\partial_\psi, \\
Y_2 &= X_2 + \frac{1}{4}\left(2\cos(\omega t)\omega + i\sin(\omega t)\omega^2(x^2+y^2)\right)\partial_\psi, \\
Y_3 &= X_3, \quad Y_4 = X_4, \quad Y_5 = X_5 - \frac{1}{2}\omega\left(x\cos(\omega t) + y\sin(\omega t)\right)\partial_\psi, \\
Y_6 &= X_6 + \frac{1}{2}\omega\left(x\sin(\omega t) - y\cos(\omega t)\right)\partial_\psi, \quad Y_7 = X_7 + \frac{i}{2}\omega x\partial_\psi, \\
Y_8 &= X_8 - \frac{i}{2}\omega y\partial_\psi, \qquad (7)
\end{aligned}$$

plus $\psi\partial_\psi$ and $\alpha(t,x,y)\partial_\psi$ s.t. α satisfies (6).

Quantizing with Noether: Act 2

MCN, J. Nonlinear Math.Phys., 2013

Quantizing with Noether: Act 2

MCN, J. Nonlinear Math.Phys., 2013

- Find the Lie symmetries of the Lagrange equations

$$\Upsilon = W(t, \underline{x})\partial_t + \sum_{k=1}^{N} W_k(t, \underline{x})\partial_{x_k}$$

Quantizing with Noether: Act 2

MCN, J. Nonlinear Math.Phys., 2013

- Find the Lie symmetries of the Lagrange equations

$$\Upsilon = W(t, \underline{x})\partial_t + \sum_{k=1}^{N} W_k(t, \underline{x})\partial_{x_k}$$

- Find the Noether symmetries

$$\Gamma = V(t, \underline{x})\partial_t + \sum_{k=1}^{N} V_k(t, \underline{x})\partial_{x_k}, \quad \Gamma \subset \Upsilon$$

Quantizing with Noether: Act 2

MCN, J. Nonlinear Math.Phys., 2013

- Find the Lie symmetries of the Lagrange equations

$$\Upsilon = W(t,\underline{x})\partial_t + \sum_{k=1}^{N} W_k(t,\underline{x})\partial_{x_k}$$

- Find the Noether symmetries

$$\Gamma = V(t,\underline{x})\partial_t + \sum_{k=1}^{N} V_k(t,\underline{x})\partial_{x_k}, \quad \Gamma \subset \Upsilon$$

- Construct the Schrödinger equation admitting those Lie symmetries

$$2i\psi_t + \sum_{k,j=1}^{N} f_{kj}(\underline{x})\psi_{x_j x_k} + \sum_{k=1}^{N} h_k(\underline{x})\psi_{x_k} + f_3(\underline{x})\psi = 0$$

$$\Omega = V(t,\underline{x})\partial_t + \sum_{k=1}^{N} V_k(t,\underline{x})\partial_{x_k} + G(t,\underline{x},\psi)\partial_\psi$$

Quantizing with Noether: Act 2

MCN, J. Nonlinear Math.Phys., 2013

- Find the Lie symmetries of the Lagrange equations

$$\Upsilon = W(t, \underline{x})\partial_t + \sum_{k=1}^{N} W_k(t, \underline{x})\partial_{x_k}$$

- Find the Noether symmetries

$$\Gamma = V(t, \underline{x})\partial_t + \sum_{k=1}^{N} V_k(t, \underline{x})\partial_{x_k}, \quad \Gamma \subset \Upsilon$$

- Construct the Schrödinger equation admitting those Lie symmetries

$$2i\psi_t + \sum_{k,j=1}^{N} f_{kj}(\underline{x})\psi_{x_j x_k} + \sum_{k=1}^{N} h_k(\underline{x})\psi_{x_k} + f_3(\underline{x})\psi = 0$$

$$\Omega = V(t, \underline{x})\partial_t + \sum_{k=1}^{N} V_k(t, \underline{x})\partial_{x_k} + G(t, \underline{x}, \psi)\partial_\psi$$

Quantizing with Noether: Act 2

MCN, J. Nonlinear Math.Phys., 2013

- Find the Lie symmetries of the Lagrange equations

$$\Upsilon = W(t,\underline{x})\partial_t + \sum_{k=1}^{N} W_k(t,\underline{x})\partial_{x_k}$$

- Find the Noether symmetries

$$\Gamma = V(t,\underline{x})\partial_t + \sum_{k=1}^{N} V_k(t,\underline{x})\partial_{x_k}, \quad \Gamma \subset \Upsilon$$

- Construct the Schrödinger equation admitting those Lie symmetries

$$2i\psi_t + \sum_{k,j=1}^{N} f_{kj}(\underline{x})\psi_{x_j x_k} + \sum_{k=1}^{N} h_k(\underline{x})\psi_{x_k} + f_3(\underline{x})\psi = 0$$

$$\Omega = V(t, \ldots)\partial \ldots + \sum^{N} V_k(t, \ldots)\partial \ldots + G(t, \ldots)\partial \ldots$$

QUANTIZE PRESERVING THE SYMMETRIES!

N charged particles in a uniform magnetic field

MCN, Theor.Math.Phys., 2016 In complex variables the Newtonian eqs are:

$$\ddot{r}_k = i\omega \dot{r}_k, \quad (k = 1, \ldots, N).$$

N charged particles in a uniform magnetic field

MCN, Theor.Math.Phys., 2016 In complex variables the Newtonian eqs are:

$$\ddot{r}_k = i\omega\dot{r}_k, \quad (k = 1, \ldots, N).$$

Among many Lagrangians, let us consider the following:

$$L_0 = \frac{\exp(-i\omega t)}{2} \sum_{k=1}^{N} \dot{r}_k^2,$$

that admits $(N^2 + 3N + 6)/2$ Noether symmetries.

N charged particles in a uniform magnetic field

MCN, Theor.Math.Phys., 2016 In complex variables the Newtonian eqs are:

$$\ddot{r}_k = i\omega\dot{r}_k, \quad (k = 1, \ldots, N).$$

Among many Lagrangians, let us consider the following:

$$L_0 = \frac{\exp(-i\omega t)}{2}\sum_{k=1}^{N}\dot{r}_k^2,$$

that admits $(N^2 + 3N + 6)/2$ Noether symmetries.
Yet there is another time-independent Lagrangian:

$$L = \sum_{k=1}^{N}\left(\frac{1}{2}\dot{r}_k\dot{\tilde{r}}_k + \frac{i\omega}{4}(\tilde{r}_k\dot{r}_k - r_k\dot{\tilde{r}}_k)\right)$$

that also admits $(N^2 + 3N + 6)/2$ Noether symmetries.

Replacing

$$r_k = x_k + iy_k, \quad \tilde{r}_k = x_k - iy_k$$

into the Lagrangian L one gets:

$$L = \sum_{k=1}^{N} \left(\frac{1}{2}(\dot{x}_k^2 + \dot{y}_k^2) + \frac{\omega}{2}(y_k \dot{x}_k - x_k \dot{y}_k) \right),$$

and the Lagrangian equations are:

$$\ddot{x}_k = -\omega \dot{y}_k, \quad \ddot{y}_k = \omega \dot{x}_k.$$

Replacing

$$r_k = x_k + iy_k, \quad \widetilde{r}_k = x_k - iy_k$$

into the Lagrangian L one gets:

$$L = \sum_{k=1}^{N} \left(\frac{1}{2}(\dot{x}_k^2 + \dot{y}_k^2) + \frac{\omega}{2}(y_k \dot{x}_k - x_k \dot{y}_k) \right),$$

and the Lagrangian equations are:

$$\ddot{x}_k = -\omega \dot{y}_k, \quad \ddot{y}_k = \omega \dot{x}_k.$$

Then the Schrödinger equation that can be obtained by preserving the symmetries is:

$$2i\psi_t + \sum_{k=1}^{N} \left(\psi_{x_k x_k} + \psi_{y_k y_k} + i\omega(y_k \psi_{x_k} - x_k \psi_{y_k}) - \frac{\omega^2}{4}(x_k^2 + y_k^2)\psi \right) = 0.$$

Replacing

$$r_k = x_k + iy_k, \quad \widetilde{r}_k = x_k - iy_k$$

into the Lagrangian L one gets:

$$L = \sum_{k=1}^{N} \left(\frac{1}{2}(\dot{x}_k^2 + \dot{y}_k^2) + \frac{\omega}{2}(y_k \dot{x}_k - x_k \dot{y}_k) \right),$$

and the Lagrangian equations are:

$$\ddot{x}_k = -\omega \dot{y}_k, \quad \ddot{y}_k = \omega \dot{x}_k.$$

Then the Schrödinger equation that can be obtained by preserving the symmetries is:

$$2i\psi_t + \sum_{k=1}^{N} \left(\psi_{x_k x_k} + \psi_{y_k y_k} + i\omega(y_k \psi_{x_k} - x_k \psi_{y_k}) - \frac{\omega^2}{4}(x_k^2 + y_k^2)\psi \right) = 0.$$

Indeed, this eq. admits a $\frac{N^2+3N+6}{2} + 1 + \infty$ dimensional Lie symmetry algebra.

A simplification: the role of $sl(N+2,\mathbb{R})$

In *G.Gubbiotti & MCN, J. Nonlinear Math.Phys., 2014*, if the N Lagrangian equations admit a (N^2+4N+3)-dim Lie symmetry algebra, $sl(N+2,\mathbb{R})$, namely they are linearizable by a point transformation, then we simplified the algorithm as follows:

A simplification: the role of $sl(N+2,\mathbb{R})$

In *G.Gubbiotti & MCN, J. Nonlinear Math.Phys., 2014*, if the N Lagrangian equations admit a (N^2+4N+3)-dim Lie symmetry algebra, $sl(N+2,\mathbb{R})$, namely they are linearizable by a point transformation, then we simplified the algorithm as follows:

- Find the linearizing transformation.

A simplification: the role of $sl(N+2,\mathbb{R})$

In *G.Gubbiotti & MCN, J. Nonlinear Math.Phys., 2014*, if the N Lagrangian equations admit a (N^2+4N+3)-dim Lie symmetry algebra, $sl(N+2,\mathbb{R})$, namely they are linearizable by a point transformation, then we simplified the algorithm as follows:

- Find the linearizing transformation.
- Derive the Lagrangian that admits the maximum number of Noether symmetries by applying the linearizing transformation to the natural Lagrangian of the corresponding linear system.

A simplification: the role of $sl(N+2,\mathbb{R})$

In *G.Gubbiotti & MCN, J. Nonlinear Math.Phys., 2014*, if the N Lagrangian equations admit a (N^2+4N+3)-dim Lie symmetry algebra, $sl(N+2,\mathbb{R})$, namely they are linearizable by a point transformation, then we simplified the algorithm as follows:

- Find the linearizing transformation.
- Derive the Lagrangian that admits the maximum number of Noether symmetries by applying the linearizing transformation to the natural Lagrangian of the corresponding linear system.
- Derive the Schrödinger equation by applying the linearizing transformation to the Schrödinger equation of the corresponding linear system.

A simplification: the role of $sl(N+2,\mathbb{R})$

In *G.Gubbiotti & MCN, J. Nonlinear Math.Phys., 2014*, if the N Lagrangian equations admit a (N^2+4N+3)-dim Lie symmetry algebra, $sl(N+2,\mathbb{R})$, namely they are linearizable by a point transformation, then we simplified the algorithm as follows:

- Find the linearizing transformation.
- Derive the Lagrangian that admits the maximum number of Noether symmetries by applying the linearizing transformation to the natural Lagrangian of the corresponding linear system.
- Derive the Schrödinger equation by applying the linearizing transformation to the Schrödinger equation of the corresponding linear system.

See also *G. Gubbiotti & MCN, J.Math.Anal.Appl., 2015*.

Harmonic oscillator on a double cone

G.Gubbiotti & MCN, J. Nonlinear Math.Phys., 2017

N.B.: $m = 1$, opening angle $2\alpha, \alpha \in \left(0, \frac{\pi}{2}\right)$, $k = \sin(\alpha)$

The Lagrangian $L_{\text{ho}} = \frac{1}{2}\left(\dot{r}^2 + k^2 r^2 \dot{\phi}^2\right) - \frac{1}{2}\omega^2 r^2$ admits 8 Noether symmetries, and the corresponding Lagrangian equations

$$\ddot{r} = k^2 r \dot{\phi}^2 - \omega^2 r, \quad \ddot{\phi} = -2\frac{\dot{r}\dot{\phi}}{r} \qquad (\spadesuit)$$

admit a 15-dim Lie symmetry algebra.

Harmonic oscillator on a double cone

G.Gubbiotti & MCN, J. Nonlinear Math.Phys., 2017

N.B.: $m = 1$, opening angle $2\alpha, \alpha \in \left(0, \frac{\pi}{2}\right)$, $k = \sin(\alpha)$

The Lagrangian $L_{\text{ho}} = \frac{1}{2}\left(\dot{r}^2 + k^2 r^2 \dot{\phi}^2\right) - \frac{1}{2}\omega^2 r^2$ admits 8 Noether symmetries, and the corresponding Lagrangian equations

$$\ddot{r} = k^2 r \dot{\phi}^2 - \omega^2 r, \quad \ddot{\phi} = -2\frac{\dot{r}\dot{\phi}}{r} \qquad (\spadesuit)$$

admit a 15-dim Lie symmetry algebra. Then, the transformation

$$u = r\cos(k\phi), \quad v = r\sin(k\phi) \qquad (\diamondsuit)$$

yields:

$$\ddot{u} + \omega^2 u = 0, \quad \ddot{v} + \omega^2 v = 0,$$

a two-dim harmonic oscillator in (u, v).

Harmonic oscillator on a double cone

G.Gubbiotti & MCN, J. Nonlinear Math.Phys., 2017

N.B.: $m = 1$, opening angle $2\alpha, \alpha \in \left(0, \frac{\pi}{2}\right)$, $k = \sin(\alpha)$

The Lagrangian $L_{\text{ho}} = \frac{1}{2}\left(\dot{r}^2 + k^2 r^2 \dot{\phi}^2\right) - \frac{1}{2}\omega^2 r^2$ admits 8 Noether symmetries, and the corresponding Lagrangian equations

$$\ddot{r} = k^2 r \dot{\phi}^2 - \omega^2 r, \quad \ddot{\phi} = -2\frac{\dot{r}\dot{\phi}}{r} \qquad (\spadesuit)$$

admit a 15-dim Lie symmetry algebra. Then, the transformation

$$u = r\cos(k\phi), \quad v = r\sin(k\phi) \qquad (\diamondsuit)$$

yields:

$$\ddot{u} + \omega^2 u = 0, \quad \ddot{v} + \omega^2 v = 0,$$

a two-dim harmonic oscillator in (u, v). If we apply the linearizing transformation $(\diamondsuit)$ to: $2\mathrm{i}\psi_t + \psi_{uu} + \psi_{vv} - \omega^2(u^2 + v^2)\psi = 0,$

Harmonic oscillator on a double cone

G.Gubbiotti & MCN, J. Nonlinear Math.Phys., 2017

N.B.: $m = 1$, opening angle $2\alpha, \alpha \in \left(0, \frac{\pi}{2}\right)$, $k = \sin(\alpha)$

The Lagrangian $L_{\text{ho}} = \frac{1}{2}\left(\dot{r}^2 + k^2 r^2 \dot{\phi}^2\right) - \frac{1}{2}\omega^2 r^2$ admits 8 Noether symmetries, and the corresponding Lagrangian equations

$$\ddot{r} = k^2 r \dot{\phi}^2 - \omega^2 r, \quad \ddot{\phi} = -2\frac{\dot{r}\dot{\phi}}{r} \qquad (\spadesuit)$$

admit a 15-dim Lie symmetry algebra. Then, the transformation

$$u = r\cos(k\phi), \quad v = r\sin(k\phi) \qquad (\diamondsuit)$$

yields:

$$\ddot{u} + \omega^2 u = 0, \quad \ddot{v} + \omega^2 v = 0,$$

a two-dim harmonic oscillator in (u, v). If we apply the linearizing transformation $(\diamondsuit)$ to: $2\mathrm{i}\psi_t + \psi_{uu} + \psi_{vv} - \omega^2(u^2 + v^2)\psi = 0$, then the Schrödinger equation corresponding to system $(\spadesuit)$ is

$$2\mathrm{i}\psi_t + \psi_{rr} + \frac{1}{r}\psi_r + \frac{1}{k^2 r^2}\psi_{\phi\phi} - \omega^2 r^2 \psi = 0.$$

Noether symmetries of the Lagrangian:

$$L_{\text{ho}} = \frac{1}{2}\left(\dot{r}^2 + k^2 r^2 \dot{\phi}^2\right) - \frac{1}{2}\omega^2 r^2$$

$$\Gamma_8 = \partial_\phi,\ \Gamma_9 = \partial_t,\ \Gamma_{10} = \cos(2\omega t)\partial_t - \omega\sin(2\omega t) r\partial_r,$$
$$\Gamma_{11} = \sin(2\omega t)\partial_t + \omega\cos(2\omega t) r\partial_r,$$
$$\Gamma_{12} = \cos(\omega t)\left(\cos(k\phi)\partial_r - \frac{1}{kr}\sin(k\phi)\partial_\phi\right),$$
$$\Gamma_{13} = \sin(\omega t)\left(\cos(k\phi)\partial_r - \frac{1}{kr}\sin(k\phi)\partial_\phi\right),$$
$$\Gamma_{14} = \cos(\omega t)\left(\sin(k\phi)\partial_r + \frac{1}{kr}\cos(k\phi)\partial_\phi\right),$$
$$\Gamma_{15} = \sin(\omega t)\left(\sin(k\phi)\partial_r + \frac{1}{kr}\cos(k\phi)\partial_\phi\right).$$

Noether symmetries of the Lagrangian:

$$L_{\text{ho}} = \frac{1}{2}\left(\dot{r}^2 + k^2 r^2 \dot{\phi}^2\right) - \frac{1}{2}\omega^2 r^2$$

$\Gamma_8 = \partial_\phi,\ \Gamma_9 = \partial_t,\ \Gamma_{10} = \cos(2\omega t)\partial_t - \omega\sin(2\omega t) r\partial_r,$
$\Gamma_{11} = \sin(2\omega t)\partial_t + \omega\cos(2\omega t) r\partial_r,$
$\Gamma_{12} = \cos(\omega t)\left(\cos(k\phi)\partial_r - \frac{1}{kr}\sin(k\phi)\partial_\phi\right),$
$\Gamma_{13} = \sin(\omega t)\left(\cos(k\phi)\partial_r - \frac{1}{kr}\sin(k\phi)\partial_\phi\right),$
$\Gamma_{14} = \cos(\omega t)\left(\sin(k\phi)\partial_r + \frac{1}{kr}\cos(k\phi)\partial_\phi\right),$
$\Gamma_{15} = \sin(\omega t)\left(\sin(k\phi)\partial_r + \frac{1}{kr}\cos(k\phi)\partial_\phi\right).$

Lie symmetries of the Schrödinger equation:

$$2\mathrm{i}\psi_t + \psi_{rr} + \frac{1}{r}\psi_r + \frac{1}{k^2 r^2}\psi_{\phi\phi} - \omega^2 r^2 \psi = 0$$

$\Omega_1 = \Gamma_8, \Omega_2 = \Gamma_9, \Omega_3 = \Gamma_{10} + \omega\left(\sin(2\omega t) - 2\mathrm{i}\cos(2\omega t)\omega r^2\right)\psi\partial_\psi,$
$\Omega_4 = \Gamma_{11} - \omega\left(\cos(2\omega t) + 2\mathrm{i}\sin(2\omega t)\omega r^2\right)\psi\partial_\psi,$
$\Omega_5 = \Gamma_{12} - \mathrm{i}\omega r\sin(\omega t)\cos(k\phi)\psi\partial_\psi,$
$\Omega_6 = \Gamma_{13} + \mathrm{i}\omega r\cos(\omega t)\cos(k\phi)\psi\partial_\psi,$
$\Omega_7 = \Gamma_{14} - \mathrm{i}\omega r\sin(\omega t)\sin(k\phi)\psi\partial_\psi,$
$\Omega_8 = \Gamma_{15} + \mathrm{i}\omega r\cos(\omega t)\cos(k\phi)\psi\partial_\psi.$

Who is right?

K. Kowalski, J. Rembieliński, Annals of Physics, 2013 took the usual angular momentum $\hat{p}_\phi = -\mathrm{i}\partial_\phi$ and looked for self-adjoint operators of the type $\hat{p}_r = -\mathrm{i}\left(\partial_r + F(r)\right)$, with respect to the scalar product $\langle f, g\rangle = \int_0^{2\pi}\int_{-\infty}^{\infty} f^* g\,|r|\,\mathrm{d}r\mathrm{d}\phi$, on the space of square integrable functions $f(r,\phi), g(r,\phi)$ on the cone. This yields that the self-adjoint operator is $\hat{p}_r = -\mathrm{i}\left(\partial_r + \frac{1}{2r}\right)$, and consequently their Schrödinger equation is:

$$2\mathrm{i}\psi_t + \psi_{rr} + \frac{1}{r}\psi_r + \frac{1}{4k^2r^2}\psi_{\phi\phi} - \left(\frac{1}{4r^2} + \omega^2 r^2\right)\psi = 0.$$

The additional term $-\frac{\psi}{4r^2}$ breaks the symmetries, i.e. four out of eight symmetries are not preserved.

Also ($p \in \mathbb{Z}$):

$E_n = \omega\left(2n + \frac{1}{2}\sqrt{1 + \frac{4p^2}{k^2}} + 1\right)$ vs. $E_n = \omega\left(2n + \frac{|p|}{k} + 1\right)$.

Further insight is needed especially from the experimentalists.

However...

DeWitt's approach

Introducing the self-adjoint momentum operators as defined in *Bryce Seligman DeWitt, Phys. Rev., 1952*, i.e.:

$$\hat{p}_k = -\mathrm{i}\left(\frac{\partial}{\partial_{q_k}} + \Gamma^j_{kj}\right)$$

with Γ^j_{jk} the contracted Christoffel symbols, yield the same angular $\hat{p}_\phi = -\mathrm{i}\partial_\phi$ and radial momenta $\hat{p}_r = -\mathrm{i}\left(\partial_r + \frac{1}{2r}\right)$ given by K&R.

DeWitt's approach

Introducing the self-adjoint momentum operators as defined in *Bryce Seligman DeWitt, Phys. Rev., 1952*, i.e.:

$$\hat{p}_k = -\mathrm{i}\left(\frac{\partial}{\partial_{q_k}} + \Gamma^j_{kj}\right)$$

with Γ^j_{jk} the contracted Christoffel symbols, yield the same angular $\hat{p}_\phi = -\mathrm{i}\partial_\phi$ and radial momenta $\hat{p}_r = -\mathrm{i}\left(\partial_r + \frac{1}{2r}\right)$ given by K&R.
However, the Hamiltonian operator shall include an extra quantum mechanical potential Q, *"which must be added to the covariant classical Hamiltonian in order to produce the covariant quantum Hamiltonian"* as stated by DeWitt.
Adding this potential Q means to eliminate the symmetry-breaking term $-\psi/4r^2$, and consequently

DeWitt's approach

Introducing the self-adjoint momentum operators as defined in *Bryce Seligman DeWitt, Phys. Rev., 1952*, i.e.:

$$\hat{p}_k = -\mathrm{i}\left(\frac{\partial}{\partial_{q_k}} + \Gamma^j_{kj}\right)$$

with Γ^j_{jk} the contracted Christoffel symbols, yield the same angular $\hat{p}_\phi = -\mathrm{i}\partial_\phi$ and radial momenta $\hat{p}_r = -\mathrm{i}\left(\partial_r + \frac{1}{2r}\right)$ given by K&R.
However, the Hamiltonian operator shall include an extra quantum mechanical potential Q, *"which must be added to the covariant classical Hamiltonian in order to produce the covariant quantum Hamiltonian"* as stated by DeWitt.
Adding this potential Q means to eliminate the symmetry-breaking term $-\psi/4r^2$, and consequently **the quantization method that preserves the Noether symmetries of the classical problem corresponds to DeWitt's approach.**

Noether vs Schrödinger

Applying Noether's theorem yields that L_{13}, L_{62} and L_{87} admit five Noether point symmetries. The main difference among the three Lagrangians is that they admit different Noether point symmetries. Which Schrödinger-type equations are obtained??:

$$L_{8,7} = \frac{1}{2}\dot{x}^2 \implies 2i\psi_t + \psi_{xx} = 0$$

Noether vs Schrödinger

Applying Noether's theorem yields that L_{13}, L_{62} and L_{87} admit five Noether point symmetries. The main difference among the three Lagrangians is that they admit different Noether point symmetries. Which Schrödinger-type equations are obtained??:

$$L_{8,7} = \frac{1}{2}\dot{x}^2 \implies 2i\psi_t + \psi_{xx} = 0$$

$$L_{6,2} = \frac{1}{2\dot{x}} \implies$$

Noether vs Schrödinger

Applying Noether's theorem yields that L_{13}, L_{62} and L_{87} admit five Noether point symmetries. The main difference among the three Lagrangians is that they admit different Noether point symmetries. Which Schrödinger-type equations are obtained??:

$$L_{8,7} = \frac{1}{2}\dot{x}^2 \implies 2i\psi_t + \psi_{xx} = 0$$

$$L_{6,2} = \frac{1}{2\dot{x}} \implies 2i\psi_x + \psi_{tt} = 0$$

Noether vs Schrödinger

Applying Noether's theorem yields that L_{13}, L_{62} and L_{87} admit five Noether point symmetries. The main difference among the three Lagrangians is that they admit different Noether point symmetries. Which Schrödinger-type equations are obtained??:

$$L_{8,7} = \frac{1}{2}\dot{x}^2 \implies 2i\psi_t + \psi_{xx} = 0$$

$$L_{6,2} = \frac{1}{2\dot{x}} \implies 2i\psi_x + \psi_{tt} = 0$$

$$L_{1,3} = \frac{1}{2t^2(x - t\dot{x})} \implies$$

Noether vs Schrödinger

Applying Noether's theorem yields that L_{13}, L_{62} and L_{87} admit five Noether point symmetries. The main difference among the three Lagrangians is that they admit different Noether point symmetries. Which Schrödinger-type equations are obtained??:

$$L_{8,7} = \frac{1}{2}\dot{x}^2 \implies 2i\psi_t + \psi_{xx} = 0$$

$$L_{6,2} = \frac{1}{2\dot{x}} \implies 2i\psi_x + \psi_{tt} = 0$$

$$L_{1,3} = \frac{1}{2t^2(x - t\dot{x})} \implies t^2\psi_{tt} + 2tx\psi_{tx} + x^2\psi_{xx} = 0$$

$$\Rightarrow$$

Noether vs Schrödinger

Applying Noether's theorem yields that L_{13}, L_{62} and L_{87} admit five Noether point symmetries. The main difference among the three Lagrangians is that they admit different Noether point symmetries. Which Schrödinger-type equations are obtained??:

$$L_{8,7} = \frac{1}{2}\dot{x}^2 \quad \Longrightarrow \quad 2i\psi_t + \psi_{xx} = 0$$

$$L_{6,2} = \frac{1}{2\dot{x}} \quad \Longrightarrow \quad 2i\psi_x + \psi_{tt} = 0$$

$$L_{1,3} = \frac{1}{2t^2(x - t\dot{x})} \quad \Longrightarrow \quad t^2\psi_{tt} + 2tx\psi_{tx} + x^2\psi_{xx} = 0$$

$$\Rightarrow \xi = \frac{x}{t} \Rightarrow$$

Noether vs Schrödinger

Applying Noether's theorem yields that L_{13}, L_{62} and L_{87} admit five Noether point symmetries. The main difference among the three Lagrangians is that they admit different Noether point symmetries. Which Schrödinger-type equations are obtained??:

$$L_{8,7} = \frac{1}{2}\dot{x}^2 \quad \Longrightarrow \quad 2i\psi_t + \psi_{xx} = 0$$

$$L_{6,2} = \frac{1}{2\dot{x}} \quad \Longrightarrow \quad 2i\psi_x + \psi_{tt} = 0$$

$$L_{1,3} = \frac{1}{2t^2(x - t\dot{x})} \quad \Longrightarrow \quad t^2\psi_{tt} + 2tx\psi_{tx} + x^2\psi_{xx} = 0$$

$$\Rightarrow \xi = \frac{x}{t} \Rightarrow \frac{\partial^2\psi}{\partial t^2}(t, \xi) = 0.$$

Noether vs Schrödinger

Applying Noether's theorem yields that L_{13}, L_{62} and L_{87} admit five Noether point symmetries. The main difference among the three Lagrangians is that they admit different Noether point symmetries. Which Schrödinger-type equations are obtained??:

$$L_{8,7} = \frac{1}{2}\dot{x}^2 \quad \Longrightarrow \quad 2i\psi_t + \psi_{xx} = 0$$

$$L_{6,2} = \frac{1}{2\dot{x}} \quad \Longrightarrow \quad 2i\psi_x + \psi_{tt} = 0$$

$$L_{1,3} = \frac{1}{2t^2(x - t\dot{x})} \quad \Longrightarrow \quad t^2\psi_{tt} + 2tx\psi_{tx} + x^2\psi_{xx} = 0$$

$$\Rightarrow \xi = \frac{x}{t} \Rightarrow \frac{\partial^2\psi}{\partial t^2}(t,\xi) = 0.$$

EITHER INVERT TIME WITH SPACE
OR GO BACK TO CLASSICAL MECHANICS !!!

Quantizing with Noether: Final Act (?)

Quantizing with Noether: Final Act (?)

- Find the Lie symmetries of the Lagrange equations

$$\Upsilon = W(t, \underline{x})\partial_t + \sum_{k=1}^{N} W_k(t, \underline{x})\partial_{x_k}$$

Quantizing with Noether: Final Act (?)

- Find the Lie symmetries of the Lagrange equations

$$\Upsilon = W(t,\underline{x})\partial_t + \sum_{k=1}^{N} W_k(t,\underline{x})\partial_{x_k}$$

- Find the right Noether symmetries

$$\Gamma = V(t,\underline{x})\partial_t + \sum_{k=1}^{N} V_k(t,\underline{x})\partial_{x_k}, \quad \Gamma \subset \Upsilon$$

Quantizing with Noether: Final Act (?)

- Find the Lie symmetries of the Lagrange equations

$$\Upsilon = W(t,\underline{x})\partial_t + \sum_{k=1}^{N} W_k(t,\underline{x})\partial_{x_k}$$

- Find the right Noether symmetries

$$\Gamma = V(t,\underline{x})\partial_t + \sum_{k=1}^{N} V_k(t,\underline{x})\partial_{x_k}, \quad \Gamma \subset \Upsilon$$

- Construct the Schrödinger equation admitting those Lie symmetries

$$2iu_t + \sum_{k,j=1}^{N} f_{kj}(\underline{x})u_{x_j x_k} + \sum_{k=1}^{N} h_k(\underline{x})u_{x_k} + f_3(\underline{x})u = 0$$

$$\Omega = V(t,\underline{x})\partial_t + \sum_{k=1}^{N} V_k(t,\underline{x})\partial_{x_k} + G(t,\underline{x},u)\partial_u$$

Quantizing with Noether: Final Act (?)

- Find the Lie symmetries of the Lagrange equations

$$\Upsilon = W(t,\underline{x})\partial_t + \sum_{k=1}^{N} W_k(t,\underline{x})\partial_{x_k}$$

- Find the <u>right</u> Noether symmetries

$$\Gamma = V(t,\underline{x})\partial_t + \sum_{k=1}^{N} V_k(t,\underline{x})\partial_{x_k}, \quad \Gamma \subset \Upsilon$$

- Construct the Schrödinger equation admitting those Lie symmetries

$$2iu_t + \sum_{k,j=1}^{N} f_{kj}(\underline{x})u_{x_j x_k} + \sum_{k=1}^{N} h_k(\underline{x})u_{x_k} + f_3(\underline{x})u = 0$$

$$\Omega = V(t,\underline{x})\partial_t + \sum_{k=1}^{N} V_k(t,\underline{x})\partial_{x_k} + G(t,\underline{x},u)\partial_u$$

QUANTIZE PRESERVING THE RIGHT SYMMETRIES!

Quantizing with Noether: Final Act (?)

- Find the Lie symmetries of the Lagrange equations

$$\Upsilon = W(t,\underline{x})\partial_t + \sum_{k=1}^{N} W_k(t,\underline{x})\partial_{x_k}$$

- Find the right Noether symmetries

$$\Gamma = V(t,\underline{x})\partial_t + \sum_{k=1}^{N} V_k(t,\underline{x})\partial_{x_k}, \quad \Gamma \subset \Upsilon$$

- Construct the Schrödinger equation admitting those Lie symmetries

$$2iu_t + \sum_{k,j=1}^{N} f_{kj}(\underline{x})u_{x_j x_k} + \sum_{k=1}^{N} h_k(\underline{x})u_{x_k} + f_3(\underline{x})u = 0$$

$$\Omega = V(t,\underline{x})\partial_t + \sum^{N} V_k(t,\underline{x})\partial_{x_k} + G(t,\underline{x},u)\partial_u$$

QUANTIZE PRESERVING THE RIGHT REPRESENTATION!